**Non-separability**

We’ve discussed how to solve separable ODEs, which are those ODEs which we can put into the form

\[ h(y)y' = g(x) \]

or, equivalently,

\[ -g(x) \, dx + h(y) \, dy = 0. \]

It is not always obvious when this is possible, and when it is not. For instance

\[ y' = xy - y - x + 1 \]

is separable because we can write \( xy - y - x + 1 = (x - 1)(y - 1) \), and then we have (for \( y \neq 1 \))

\[ \frac{y'}{y - 1} = x - 1. \]

On the other hand,

\[ y' = x + y \quad (1) \]

is not separable. Since we will shortly develop more sophisticated methods to handle equations which are not separable, this doesn’t pose much of a concern, however the question remains - how do we know that (1) is not separable?

We can prove this is the case by assuming it were separable, and deriving a contradiction. For instance, if it were separable, we would be able to write

\[ x + y = \frac{g(x)}{h(y)} \]

for some functions \( g(x) \) and \( h(y) \). Suppose for the moment that \( h(0) \) is defined, and nonzero. Then by substituting \( y = 0 \) we would find

\[ x = \frac{g(x)}{h(0)} \implies g(x) = h(0)x, \]

and therefore

\[ x + y = \frac{h(0)x}{h(y)}. \]

Now, if we substitute \( x = 0 \), we would find

\[ y = 0, \]

which should hold for all \( y \), however is clearly a contradiction for any \( y \neq 0 \). We had assumed \( h(0) \) was defined and nonzero to make this work, but the same technique works as long as \( h(c) \) is defined and nonzero for some value \( c \), in which case we just substitute \( y = c \) instead.

This entire idea can be generalized and cleaned up in the following theorem

\[ ^* \text{We could interpret this result as saying } \]
**Theorem 1.** (Non-Separability Test) Consider the IVP
\[ y' = f(x, y), \quad y(x_0) = y_0, \]
where \( f(x, y) \) is continuous near \((x_0, y_0)\). If
\[ f(x_0, y_0)f(x, y_0) \neq f(x_0, y_0)f(x, y), \]
this cannot be solved as a separable equation.

**Proof.** We prove this by contrapositive - suppose it is separable. Then
\( f(x, y) = g(x)/h(y) \) must be defined for \((x_0, y_0)\), and so
\[
f(x_0, y_0)f(x, y_0) = \frac{g(x_0)g(x)}{h(y_0)h(y)} = \frac{g(x_0)g(x)}{h(y_0)h(y)} = f(x_0, y_0)f(x, y).
\]
\[\square\]

Let's see how this helps us with the general question of whether \( y' = x + y \) is separable. Since we aren't given an initial condition, we are looking for the general solution, but that's no problem - we will just assign an arbitrary initial condition \( y(x_0) = y_0 \), and see for which \( x_0 \) and \( y_0 \) we can rule out separability by using the above test.

We calculate:
\[
f(x_0, y_0)f(x, y_0) = (x_0 + y)(x + y_0) = x_0x + xy + x_0y_0 + yy_0 \\
f(x_0, y_0)f(x, y) = (x_0 + y_0)(x + y) = x_0x + y_0x + x_0y + y_0y
\]

Thus, these are not equal unless \( xy + x_0y_0 = y_0x + x_0y \), or equivalently \((x - x_0)(y - y_0) = 0\). This only holds when \( x = x_0 \) or \( y = y_0 \). This implies that either the domain of our solution is one point \((x_0)\), or our solution is the constant function \( y = y_0 \). The former is not a differentiable solution, and the latter is not a solution at all, since in that case \( y'(x) = 0 \neq x + y_0 \). Hence this equation can never be solved as a separable equation.

We can even work this into a partial converse.

**Theorem 2.** (Separability Criterion) Consider the IVP
\[ y' = f(x, y), \quad y(x_0) = y_0, \]
where \( f(x, y) \) is continuous near \((x_0, y_0)\). If \( f(x_0, y_0) \neq 0 \) and
\[ f(x_0, y)f(x, y_0) = f(x_0, y_0)f(x, y), \]
this can be solved as a separable equation.
Proof. Divide (2) by \( f(x_0, y_0) \) (nonzero by hypothesis) to conclude

\[
y' = f(x, y) = \frac{f(x, y_0)f(x_0, y)}{f(x_0, y_0)}.
\]

Taking, for instance,

\[
g(x) = \frac{f(x, y_0)}{f(x_0, y_0)} \quad h(y) = \frac{1}{f(x_0, y)}
\]

yields

\[
y' = g(x)/h(y),
\]

as desired, so it is separable.

We cannot get the full converse, as the non-separable IVP

\[
y' = x(y + x), \quad y(0) = 1
\]

demonstrates, however through some rather technical arguments about the nature of continuity, we can say that

- If \( f(x, y) = 0 \) near \((x_0, y_0)\) then the equation is trivially separable as \( y' = 0 \) near \((x_0, y_0)\), and \( y(x_0) = y_0 \) is a solution.

- If \( f(x, y) \) is not identically 0 near \((x_0, y_0)\), then it is separable if and only if \( f(x_1, y)f(x, y_1) = f(x_1, y_1)f(x, y) \) for all \((x_1, y_1)\) near \((x_0, y_0)\).

We can combine everything into the following conclusion:

**Theorem 3.** (Separability) The IVP

\[
y' = f(x, y), \quad y(x_0) = y_0.
\]

where \( f(x, y) \) is continuous near \((x_0, y_0)\) is separable if and only if for all \((x_1, y_1)\) near \((x_0, y_0)\),

\[
f(x_1, y)f(x, y_1) = f(x_1, y_1)f(x, y).
\]