Laplace Transform

Here are the crucial facts to know about the Laplace Transform:

\[
\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f(t)](s) + b\mathcal{L}[g(t)](s) \quad \text{(Linearity)}
\]
\[
\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s - a) \quad \text{(Exponential Shift)}
\]
\[
\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)] - f(0) \quad \text{(Derivative Rule)}
\]
\[
\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}} \quad \text{for } \Re(s) > 0 \quad \text{(Polynomial Evaluation)}
\]

Technically, this is enough to answer any question you’ll be asked involving the Laplace Transform. Depending on how frequently you use it, it may be helpful to become familiar with the following shortcuts as well.

\[
\mathcal{L}[f(at)](s) = \frac{1}{a}\mathcal{L}[f(t)]\left(\frac{s}{a}\right) \quad \text{if } a > 0 \quad \text{(Scaling)}
\]
\[
\mathcal{L}[\sin(t)](s) = \frac{1}{s^2 + 1} \quad \text{(Sine Evaluation)}
\]
\[
\mathcal{L}[\cos(t)](s) = \frac{s}{s^2 + 1} \quad \text{(Cosine Evaluation)}
\]

Use these facts to calculate the Laplace Transform of the following expressions.

1. \( t^2 + t^3 \).
2. \( \cos 2t + 5t^8 \).
3. \( e^{4t} + te^{4t} \).
4. \( (1 + 2t + t^2)(e^{-(3+i)t} + e^{-(3-i)t}) \).
5. \( \cos^2 t \) (Hint: Use \( \cos t = \frac{e^{it} + e^{-it}}{2} \).)
6. \( t^3 \sin(2t) \) (Hint: Use \( \sin t = \frac{e^{it} - e^{-it}}{2i} \).)
7. \( (e^{3t}t + 3)^2 \).
8. \( \int_0^t u^2e^{tu} \, du \) (Hint: \( \mathcal{L}[f'(t)](s) = \ldots \))

Inverse Laplace Transform

When solving ODEs, we have a need to go backwards, i.e. given a function \( F(s) \) we ask “is there a function \( f(t) \) is such that \( \mathcal{L}[f(t)](s) = F(s) \)? The answer is known as the inverse Laplace transform, \( \mathcal{L}^{-1} \).
and properties of this are given by reading the facts from above in reverse. The most useful results are the following:

\[ L^{-1}[aF(s) + bG(s)](t) = aL^{-1}[F(s)](t) + bL^{-1}[G(s)](t) \]  
\text{(Linearity)}

\[ L^{-1}[F(s)](t) = e^{at}L^{-1}[F(s+a)](t) \]  
\text{(Exponential Shift)}

\[ L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^n}{n!} \]  
\text{(Polynomial Evaluation)}

In the following questions find the inverse Laplace transform of the given expression. (Remember, you may need to use partial fractions first.)

9. \( \frac{1}{\pi t} \).
10. \( \frac{s}{\pi^2 + 1} \).
11. \( \frac{s}{\pi^2} + \frac{2}{\pi} + \frac{12}{\pi^2} \).
12. \( \frac{2}{s^2 + 4} \).
13. \( \frac{3}{s-1} + \frac{9}{s^2 - 2s + 1} \).
14. \( \frac{s^2 + 6s^2 - 36s + 54}{s^6 - 6s^5 + 9s^4} \).
15. \( \frac{4(3s^2 - 18s + 23)}{(s^2 - 6s + 12)^2} \).

**Initial Value Problems**

Find the particular solution corresponding to the given initial conditions by using the Laplace Transform.\(^1\)

16. \( y'' - 3y' + 2 = 0, \quad y(0) = 1, \quad y'(0) = -1. \)
17. \( y'' - 5y' + 6y = e^{3t}, \quad y(0) = 2, \quad y'(0) = 1. \)
18. \( y'' + y' + y = t^2 + t, \quad y(0) = y'(0) = 1. \)
19. \( y''' - y'' + 4y' - 4 = \sin(2t), \quad y(0) = y'(0) = y''(0) = 0. \)

**Challenge Problems**

Suspiciously absent from our Laplace Transform table are functions of the form \( f(t) = t^r \), where \( r \) is an arbitrary real number. Although we have calculated \( L[t^n] = \frac{n!}{s^{n+1}} \) when \( n \in \mathbb{N} \), this does not seem to generalize to arbitrary real numbers \( r \) due to the presence of the factorial \( n! \). In this set of problems, we explore the **gamma function**, which provides just such a generalization.

Each of these problems builds on the previous problem’s results, but if you are unable to solve one don’t be discouraged! You should

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\(^1\) You can verify your solution using the Annihilator method.
feel free to move on to the next one, assuming the conclusion provided by the previous one, and fill in the details later.

The gamma function is defined as

\[ \Gamma(s) := \int_0^\infty e^{-t^s} \, dt, \quad \text{where } s > 0. \]

20. Show that \( \int_1^\infty e^{-t^s} \, dt \) converges for all real \( s \).
   (Hint: Compare with \( \int_1^\infty t^{-2} \, dt \).)

21. Show that \( \int_0^1 e^{-t^s} \, dt \) converges for all real \( s > 0 \).
   (Hint: let \( t = 1/u \), and compare the resulting integral with \( \int_1^\infty u^{-s-1} \, du \).)

22. Conclude (by the preceding exercises) that \( \int_0^\infty e^{-t^s} \, dt \) converges for all \( s > 0 \).

23. Use integration by parts to show \( \Gamma(s+1) = s\Gamma(s) \) for all \( s > 0 \).

24. Use induction (and the previous exercise) to show that \( \Gamma(n+1) = n! \) for all \( n \in \mathbb{N} \).

25. The aim of this exercise is to calculate \( \Gamma({1\over2}) \). If \( r > 0 \), let \( I(r) = \int_r^\infty e^{-u^2} \, du \).
   (a) Show that \( I^2(r) = \int_r^\infty \int_r^\infty e^{-(x^2+y^2)} \, dx \, dy \).
   (b) If \( C_1 \) and \( C_2 \) are the circular disks inscribing and circumscribing \( R = [-r,r] \times [-r,r] \), show that
   \[ \int_{C_1} e^{-(x^2+y^2)} \, dx \, dy < I^2(r) < \int_{C_2} e^{-(x^2+y^2)} \, dx \, dy. \]
   (c) Express the integrals over \( C_1 \) and \( C_2 \) using polar coordinates, and use (25b) to deduce that \( I(r) \to \sqrt{\pi} \) as \( r \to \infty \).
   (d) Use what we have done so far to prove that \( \int_0^\infty e^{-u^2} \, du = \sqrt{\pi}/2 \).
   (e) Use (25d) to deduce that \( \Gamma({1\over2}) = \sqrt{\pi} \).

26. Use (25) to calculate \( L[t^{-1/2}] \) for real \( s \).
   (Hint: to calculate \( \int_0^\infty t^{-1/2} e^{-st} \, dt = \Gamma({1\over2})/\sqrt{s} \), use a substitution \( u = st \). If you are stuck this is done in T&P on page 309.)

27. Use induction and exercise (26) to calculate \( L[t^{n-1/2}] \) for real \( s \) and \( n \in \mathbb{N} \).
   (Hint: if you are stuck this is done in T&P on page 309.)

28. Prove that, for \( r > -1 \) and real \( s \),
   \[ L[t^r] = \frac{\Gamma(r+1)}{s^{r+1}}, \quad s > 0. \]
   (Hint: Make the substitution \( u = st \) in your calculation, and use the preceding exercises.)