Exact ODEs and Integrating Factors

Find the general solution of each of the following ODEs. Some are already exact, for others you may need to find an integrating factor \( h \). Some questions have a hint as to the form the integrating factor takes. Otherwise, if it is not already exact, you should try to find an integrating factor that only depends on \( x \) (so \( h = h(x) \)) or an integrating factor that only depends on \( y \) (so \( h = h(y) \)).

1. \((y^3 + 2(x + y))\,dx + (3y^2x + 2(x + y))\,dy = 0.\)

2. \((2xy + 5 + e^{x+y})\,dx + (x^2 + e^{x+y})\,dy = 0.\)

3. \((e^{2x+y} - 1)\,dx + (e^{x+y} + e^y)\,dy = 0.\)

4. \(y(\sin(2y/x) - 1)\,dx + x(\sin(2y/x) + 1)\,dy = 0.\)

5. \(y \cos^2 y\,dx + (x \cos^2 y + 1)\,dy = 0.\)

6. \(\sqrt{1-y^2}\,dx + (\sqrt{1-y^2} + 1)\,dy = 0.\)

7. \(y(xy^2 + x^2)\,dx + x(2xy^2 + x^2)\,dy = 0.\) (Hint: use an integrating factor of the form \(x^a y^b\).)

8. \((x^2y^2 + y + 1)\,dx + x\,dy = 0.\) (Hint: use the integrating factor \((x^2y^2 + 1)^{-1}\).)

The choice of integrating factor is not unique, even up to constants, as this next example shows.

9. Consider the ODE \(e^x\,dx + (e^x + 1)\,dy = 0.\)
   (a) Show that, in it’s current form, this equation is not exact.
   (b) Show that this equation is separable, and solve it as a separable equation.
   (c) Find an integrating factor depending only on \( x \), i.e. \( h = h(x) \).
      (Hint: you did this implicitly when you solved it as a separable equation)
   (d) Find an integrating factor depending only on \( y \), i.e. \( h = h(y) \).
   (e) Check that the answers you get using each of these methods agree.

Linear First-Order ODEs

Linear First-Order ODEs, that is, those of the form

\[y' + p(x)y = q(x)\]
are, in general, not exact, however they can always be solved by using an integrating factor

\[ h(x) = e^{\int p(x) \, dx}. \]

The general solution then becomes

\[ y(x) = \frac{1}{h(x)} \int h(x)q(x) \, dx. \]

Use this to find the general solution to the following linear first-order ODEs.

10. \( y' + \frac{y}{x} = 5x^3. \)
11. \( xy' + y = x \sin x. \)
12. \( (3x + y) \, dx + x \, dy = 0. \)
13. \( y' = e^x + y. \)
14. \( y' + y = e^x \sin 2x. \)
15. \( y = 1 - e^{-x}y'. \)

Substitutions

While, in theory, we can solve every reasonable first-order ODE by finding an integrating factor, in practice finding that integrating factor can be somewhat of a challenge. In the following problems, make a \( u \)-substitution to turn the ODE into an easier, solvable ODE, and find the general solution.

16. \( y' + 1 = \log |x + y|. \)
17. \( y' = \sqrt{y^2 + x^2 - 2x}. \)
18. \( (1 - \sin(1 + x + y)) \, dx + dy = 0. \)
19. \( (\sqrt{x^2 + y + x^2 + y}) \, dx + (3(x^2 + y)x + x^2) \, dy = 0. \)

Applications

20. A cake, which was heated to 350°F, is removed from the oven and placed to cool on the counter. The thermostat in the room is set to 70°F, but it also is set to allow a 3° swing in temperature, so it fluctuates sinusoidally between 67°F and 70°F.

(a) Write an equation for the ambient temperature with respect to time, assuming at time 0 it is at 70°F.
(b) Setup a differential equation which models the temperature of the cake.
(c) Will the cake ever be 70°F?

21. A city’s population is currently 200,000, and it is estimated to double every 40 years. Recently, however, an increase in people moving into the city has added approximately 600 people per year at a uniform rate.
(a) Write a differential equation for the population of the city.
(b) How many people will be in the city in 20 years?
(c) What if the number of people moving into the city is proportional to 500,000 - P, i.e. to account for overcrowding?

22. A strain of bacteria grows at a rate proportional the amount present, and experiments indicate that they double every 24 hours, however they cannot be exposed to sunlight or they will die at a rate proportional to the amount of sunlight present.
(a) Write a differential equation for the amount of bacteria present.
Hint: The same principle as the mixing problem applies, you want to think about:

\[ B'(t) = \text{Rate of Growth} - \text{Rate of Death}. \]

It would be reasonable to model the exposure to sunlight as \( k(\cos(2\pi t) - 1) \), if time is started at midnight.
(b) Find the general solution to the ODE you derived.
(c) If, at midnight, there are 2000 bacterial cells present, and at noon there are 2500 bacterial cells present, how many will be present after 24 hours elapse?
(d) Is there a value of \( k \) (which we can interpret as the bacteria’s sensitivity to sunlight) that would actually kill all the bacteria?

Challenge Problems

23. Solve the ODE \((1 + xy + y^2) \, dx + (1 + xy + x^2) \, dy = 0\) by using an integrating factor of the form \( h = h(x + y) \).

24. When we solve a differential equation, we ideally look for a function \( y = g(x) \) so that \( x \) is the independent variable, and \( y \) is the dependent variable. Often, we have to settle for \( y \) defined implicitly in terms of \( x \) as \( f(x, y) = 0 \) however the Implicit Function Theorem from Calculus tells us that with some mild conditions on the partial derivatives of \( f \), we actually do have \( y = g(x) \), even
though we may not be able to write it down. This begs the question - might we also have $x = h(y)$ for some $y$? Perhaps more motivating, would this allow us to solve some ODEs which were otherwise outside our grasp?

(a) Recall (or research) a theorem from Multivariable Calculus that allows us to consider $x = g(y)$ in an open neighborhood of $y(y(p))$ whenever $\frac{dy}{dx}$ is continuous and nonzero at the point $x = p$.

(b) Use this idea to find a general solution to $\frac{dx}{dy} + 2yx e^{-y^2}$.
   Hint: Consider $x$ as the dependent variable.

(c) Find the particular solution for $(x - \sin y) dy + \tan y dx = 0$, where $y(1) = \frac{\pi}{6}$.

25. Let $y = f(x)$ define a curve that passes through the origin. Find a family of curves with the property that the volume of the region bounded by $y = f(x)$, $x = 0$, $x = x$ rotated about the $x$ axis is equal to the volume of the region bounded by $y = f(x)$, $y = 0$, $y = y$ rotated about the $y$ axis.

A second order linear homogeneous equation of the form

$$x^2y'' + axy' + \beta y = 0$$

where $a$ and $\beta$ are constants is called an Euler equation.

26. Show that the substitution $u = \log x$ transforms the Euler equation above to a homogeneous linear differential equation of second order with constant coefficients. Solve this equation, and undo the substitution to determine the general solution of the Euler equation.

Using the result from the previous question, solve the following Euler equations.

27. $x^2y'' - xy' - 8y = 0$.

28. $x^2y'' - 3xy' + 4y = 0$.

29. $x^2y'' - xy' + 5y = 0$. 