Applications

1. A 5 kg weight is thrown straight up from the ground with a velocity of 20 m/s. Find the velocity function \( v(t) \) if the force of air resistance is assumed to be \(-2v\). How long does the weight stay in the air? With what velocity does the weight hit the ground?\(^1\)

2. You are standing on a bridge, and drop a 1 kg rock off. It takes 3 seconds for the rock to hit the water. Assuming the force of air resistance is \(-5v\), how high is the bridge above the water?

3. An object with temperature 100 °C is placed in a medium which is kept at a constant temperature of 20 °C. After 1 minute, the temperature of the body is 80 °C. How long until the body is 30 °C?

4. A turkey is stored in the fridge, which is kept at 2 °C. It is then taken out of the fridge and placed in an oven, where the temperature is kept at a constant 180 °C. After 10 minutes, you check the meat thermometer and see that the turkey is now at 16 °C. Food should be cooked to at least 70 °C for 2 minutes to kill all bacteria.\(^2\) What is the minimum amount of time you should cook the turkey?

5. A thermometer is kept inside, where the thermostat is set to 70 °F. We then bring the thermometer outside. After the first minute, the thermometer reads 60 °F. After another minute, the thermometer reads 52.5 °F. What is the outside temperature?

Exact Equations

An ordinary differential equation

\[ P(x, y) \, dx + Q(x, y) \, dy = 0 \]

is exact if there is a function \( f(x, y) \) such that \( P = f_x \) and \( Q = f_y \).

A necessary condition for the equation to be exact is that \( P_y = Q_x \).

Determine if the following equations are exact, and, if so, solve them.

6. \( x^2 y \, dx + y^2 x \, dy = 0 \).
7. \( y^2 x \, dx + x^2 y \, dy = 0 \).
8. \( y' = \frac{-y^2 - e^{y+x}}{2xy + e^{y+x}} \).
9. \( (e^{x-y} + 1) \, dx = (e^{x-y} + 2y) \, dy \).
10. \( (\sin y + x) \, dx + \sin x \, dy = 0 \).
11. \( (3x^2 + 5y) \, dx + (5x + 5y^4 + 6y^2) \, dy = 0 \).\(^3\)

\(^1\) You could use conservation of energy to save yourself a little work here, but you do not have to.


\(^3\) Remember - the general rule of thumb is that if the solution can be written explicitly, you should write it explicitly.
12. \[
\frac{x^2y^2 + y^2 + xy + 1}{x^3y^2 + x^2y^3 + x + y} \, dx + \frac{x^2y^2 + x^2 + xy + 1}{x^3y^2 + x^2y^3 + x + y} \, dy = 0.
\]

13. The ODE \(\sin x \, dx + \cos y \, dy = 0\) is separable. Every separable equation is exact. Solve this as a separable equation, as well as an exact equation, and verify that you get the same result.

\textit{Initial Value Problems}

Find the particular solution corresponding to the given exact ODE.

14. \(2 \sin y \, dx + (2x \cos y + 2y) \, dy = 0, \quad y(0) = 1\).

15. \((x + y) \, (dx + dy) = 0 \quad y(1) = 0\).

16. \(y' = \frac{-4y - e^x}{4x} \quad y(0) = 6\).

\textit{Challenge Problems}

17. A more realistic model of air resistance is that the force is equal to \(-cv^2\) for an appropriate constant \(c\), which depends on the object. A 10 kg cannon ball is shot out of a cannon on the ground with a vertical velocity of 150 m/s. Assume \(c\) for this object is 2. Derive the equation of motion. How high does the cannon ball go?

18. Prove that if \(f(x, y) = 0\) defines \(y(x)\) implicitly, and \(f(x, y) = g(ax + by)\) for some function \(g\) with \(b \neq 0\), then \(y(x) = -\frac{g'}{b}x + c\) for some \(c\).

Note that the condition \(b \neq 0\) is not really an additional condition, since \(f(x, y) = 0\) defines \(y(x)\) implicitly.