Existence and Uniqueness

Recall that the existence theorem guarantees a solution of \( y' = f(x, y) \) with \( y(x_0) = y_0 \) if \( f(x, y) \) is continuous near \( (x_0, y_0) \). Furthermore, if \( f_y(x, y) \) is continuous near \( (x_0, y_0) \), this solution is unique. For example, if we consider the ODE

\[
(x + y)y' = 1,
\]

we are guaranteed to find a solution which exists as long as \( x_0 \neq y_0 \), since as long as \( x_0 \neq y_0 \) the function \( (x + y)^{-1} \) is continuous at \( (x_0, y_0) \). In addition, we have

\[
\frac{\partial}{\partial y}(x + y)^{-1} = -(x + y)^{-2},
\]

and this is continuous near every \( (x_0, y_0) \) such that \( x_0 \neq y_0 \), so by the uniqueness theorem we know that there is exactly one solution through each \( (x_0, y_0) \) except, perhaps for the line \( y_0 = x_0 \).

For each of the following ODE, determine the set of \( x_0 \) and \( y_0 \) for which the existence theorem guarantees a solution exists, and similarly determine the set of \( x \) and \( y \) for which the uniqueness theorem guarantees that this solution is unique.

1. \( y' = x + y \).
2. \( y' = \tan(xy) \).
3. \( x \, dx + y \, dy = 0 \).
4. \( yy' = \log(x) \).
5. \( y' = \sqrt{y^2 - 1} \).

Separation of Variables

For each ODE, determine if the equation is separable and, if so, find the general solution.

6. \( \sqrt{9 - y^2} \, dx + \sqrt{1 + x^2} \, dy = 0 \).
7. \( e^{x+y}y' = 4 \).
8. \( \sin(x + y) \, dx + \cos(x + y) \, dy = 0 \).
9. \( \sqrt{\frac{x-2}{y}} \, dx + \sqrt{x+xy} \, dy = 0 \).
10. \( y' = \frac{x-y}{y+x} \).
11. \( y' = 3x \log(xy) \).

\(^1\) By “near” \( (x_0, y_0) \), we mean in an open set containing \( (x_0, y_0) \). There are functions which are continuous at a point, but not in any open set containing that point, however this intricacy will not come up in the following questions.

Note that the existence and uniqueness theorem just tells us when we are guaranteed to have a unique solution, it doesn’t prevent us from having unique solutions at the other points.
Initial Value Problems

For the following separable ODEs:

(a) Attempt to find all particular solutions satisfying the given initial condition.
(b) Was a solution guaranteed to exist (by the existence theorem)?
(c) If (b) was yes, was it guaranteed to be unique (by the uniqueness theorem)?
(d) Compare your answers above, and explain any inconsistencies.²

12. \( yy' = 1, \quad y(0) = 0. \)
13. \( xy' = y - 1, \quad y(0) = 1. \)
14. \( y' = 2y^2x, \quad y(0) = 0. \)
15. \( y' = \frac{x}{1 - y^2}, \quad y(0) = 1. \)

Applications

For each of the following problems, set up and solve an ODE you derive from the physical system described.

16. A tank containing 200 L of secret chemical A is supposed to have an expensive and secret powder B mixed into it at a concentration of 1 kg/L, but accidentally 2 kg/L have been mixed in. Four engineers are arguing over what should be done.

"I think we should just empty the whole thing and start over," Aaron says.

"Are you crazy? That'll waste too much chemical A! Let's open the drain, and put pure chemical A in at the same rate as it is flowing out," Kylee responds.

"Ugh, that would take forever," Steve interjects. "I know, instead of putting pure chemical A in, let's add chemical A mixed with powder B at a ratio of 1 kg/L already!"

"Guys, guys - this is getting too complicated! Let's just drain out 100 L of the mix, and add 100 L of pure chemical A," Saanvi says.

(a) Which solutions actually solve the problem?
(b) Whose solution is fastest?
(c) Whose solution wastes the least chemical A?
(d) Whose solution wastes the least powder B?

² For instance, if your answer to (b) was no, but in (a) you did find a solution, why is this not a contradiction of the existence theorem? If, on the other hand, your answer to (b) was yes, but in (a) you did not find a solution, this would be a contradiction of the existence theorem, so you probably should think about it some more.
17. A tank is filled with 200 L of water. Brine with a salt concentration of 3 kg/L is added to the tank at a rate of 5 L/min. The mixture is kept uniform by stirring, and a plug is removed from the bottom so that water can flow out, also at a rate of 5 L/min.

(a) At what time will the water in the tank contain 200 kg of salt?
(b) Will it ever contain 400 kg? What about 600 kg? What about 800 kg?
(c) At what time is the rate of salt in the tank largest?

18. Bacteria grows at a rate proportional to the amount present. In ideal conditions, individual E. coli cells can double every 20 minutes. If the original culture contained 64 cells, how long will it take before there are 1 million cells? What if the original culture only contained 2 cells?

19. Radioactive materials decay at a rate proportional to the amount present. A substance’s half-life is the amount of time it takes for half the substance to decay. The half-life of carbon-14 is 5730 years.

(a) How long would it take 1 kg of carbon-14 to decay to just 10 g?
(b) A fossil is found to contain 80% less carbon than that of a living sample. How long ago did this organism die?
(c) Assuming it is difficult to measure carbon-14 accurately when less than 1% of the starting amount remains, what is the oldest fossil which can be dated using carbon dating?

Challenge Problems

20. A 400 KL pool is approximately 3% algae. The algae grows at a rate proportional to the amount present, and doubles every day. The algae will also die at a rate proportional to the amount of chlorine present. Chlorine is broken down by UV light at a rate of 0.2 L/hr.

(a) Set up a differential equation for $A(t)$, the amount of algae present at time $t$, which takes into account that the rate of algae death is proportional to $C(t)$, the amount of chlorine present.
(b) At what rate should we add Chlorine to the pool if we are to keep the total amount present at any given time constant?
(c) Assume you have a machine setup to add Chlorine at the rate deduced from (b). You also add 1 L of Chlorine manually immediately after time 0. A day later you are disappointed to find that it has only decreased the amount of algae by $\frac{1}{6}$th. How
much Chlorine should you add to completely kill the remaining algae?

(d) By manually adding Chlorine you were able to kill all the algae, but this also made the pool unusable for a while due to the high amount of Chlorine. Modify your machine from above so that it monitors the amount of algae present in the pool and instantaneously adds an appropriate amount of Chlorine necessary to keep the total algae in the pool at less than 0.1%.

21. Denote by \( y = f(t) \) the amount of a substance present at time \( t \). Assume it disintegrates at a rate proportional to the amount present. If \( n \) is a positive integer, the time \( T \) for which \( f(T) = f(0)/n \) is called the 1/nth life of the substance.

(a) Prove that the 1/nth life is the same for every sample of a given material, and compute \( T \) in terms of \( n \) and the decay constant \( k \).

(b) If \( a \) and \( b \) are given, prove that \( f \) can be expressed in the form

\[
f(t) = f(a)^{w(t)} f(b)^{1-w(t)}
\]

and determine \( w(t) \). This shows that the amount present at time \( t \) is a weighted geometric mean of the amounts present at two instants \( t = a \) and \( t = b \).