Laplace Transform of Piecewise Functions

Calculate the Laplace Transform of the following piecewise functions by rewriting them using the Heaviside function.¹

1. \( f(t) = \begin{cases} 
3 & t < 10, \\
0 & 10 < t.
\end{cases} \)

2. \( f(t) = \begin{cases} 
\frac{t^3 + 6t^2}{2} & t \leq 1, \\
e^{2t} & 1 < t \leq 2, \\
0 & 2 < t.
\end{cases} \)

3. \( f(t) = \begin{cases} 
sin 4t & t < \pi, \\
0 & \pi \leq t.
\end{cases} \)

Inverse Laplace Transform Leading to Piecewise Functions

Calculate the inverse Laplace Transform of the following functions. Rewrite the resulting function (which you should derive as some combination of Heaviside functions) as a piecewise function.

4. \( F(s) = \frac{e^{-s}}{s} \).

5. \( F(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 1)} \).

6. \( F(s) = \frac{e^{-10s}}{s^3 - 3s^2 + 3s - 1} \).

7. \( F(s) = \frac{1 + e^{-(s+1)}}{s^2 - 1} \).

IVP with Piecewise Functions

Solve the following initial value problems which involve the piecewise functions from the preceding section.

8. \( y'' + 4y' + 6y = f(t), \quad y(0) = 1, \quad y'(0) = 2, \) where
\[
f(t) = \begin{cases} 
3 & t < 10, \\
0 & 10 < t.
\end{cases}
\]

9. \( y' + y = f(t), \quad y(0) = y'(0) = 0, \) where
\[
f(t) = \begin{cases} 
\frac{t^3 + 6t^2}{2} & t \leq 1, \\
e^{2t} & 1 < t \leq 2, \\
0 & 2 < t.
\end{cases}
\]

¹ Here we use the definition of the Heaviside function we introduced in class:
\( h(t) = \begin{cases} 
0 & t < 0, \\
1 & t \geq 0.
\end{cases} \)

Recall that changing the value of a function at one point (or even making a function undefined at one point) does not change the integral of that function, and therefore does not change the Laplace Transform of that function.
10. \( y'' + 4y' + 4y = f(t), \quad y(0) = y'(0) = 0, \) where

\[
f(t) = \begin{cases} 
\sin 4t & t < \pi, \\
0 & \pi \leq t. 
\end{cases}
\]

Definition of Dirac Delta

Calculate the following integrals, which involve the Dirac Delta function.

11. \[ \int_0^{10} \delta(t - 5)(e^{t^2} + t^3 + 7 \sin t \pi) \, dt. \]
12. \[ \int_0^\infty \delta(t - \pi)(\sin(t/4) + \cos(t/3) + \cos^2 t) \, dt. \]
13. \[ \int_{-3}^3 \delta(t + 1)(te^t) \, dt. \]
14. \[ \int_{-3}^3 \delta(t - 4)(\sin t^4 + \cos t^2) \, dt. \]

IVP with Dirac Delta

Solve the following initial value problems, which involve the Dirac Delta function.

15. \( y' + 5y = \delta(t - 1), \quad y(0) = 1. \)
16. \( y'' - 16y = \delta(t - 2), \quad y(0) = y'(0) = 0. \)
17. \( y''' + 9y' = \delta(t - 1), \quad y(0) = y'(0) = 0. \)
18. \( y'' + 4y = \delta(t - \pi), \quad y(0) = 8, \quad y'(0) = 0. \)
19. \( y'' + 4y' + 5y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 3. \)

General Nonhomogeneous IVPs

Solve the following initial value problems, where \( \delta \) is the Dirac Delta function, and \( h \) is the Heaviside function.

20. \( 4y'' + 24y' + 37y = 17e^{-t} + \delta(t - \frac{1}{2}), \quad y(0) = 1, \quad y'(0) = 1. \)
21. \( y'' + 5y' + 6y = h(t - 1) + \delta(t - 2), \quad y(0) = 0, \quad y'(0) = 1. \)
22. \( y'' + 5y' + 6y = \delta(t - \frac{5}{2}) + h(t - \pi) \cos t, \quad y(0) = y'(0) = 0. \)
23. \( y'' + 4y' + 5y = (1 - h(t - 10))e^t - e^{10} \delta(t - 10), \quad y(0) = 0, \quad y'(0) = 1. \)
Challenge Problems

The Dirac Delta function has a number of useful and interesting properties, some of which are investigated in these problems.

24. Show that $\frac{d}{dt} H(t) = \delta(t)$, where $H$ is a slightly different form of the Heaviside function$^2$:

\[
H(t) = \begin{cases} 
0 & t < 0, \\
\frac{1}{2} & t = 0, \\
1 & t > 0. 
\end{cases}
\]

$^2$This version agrees with our previous definition with the exception that we previously did not define the value for $t = 0$. In particular, since the integral (and Laplace Transform) of a function is not affected by changing the value of a function at one point, all our previous results for $h$ hold for $H$. 
