Introduction to ODEs

Solutions of Differential Equations

For each of the following functions, test the proposed solution and determine whether or not it is a solution to the differential equation. Be sure you can justify your answer.

1. Is \( y = \sqrt{1 + x^2} \) a solution of \((1 + x^2)y' = xy?\)

2. Is \( y = x - 1 \) a solution of \((y - x)\,dx + dy = 0?\)

3. Is \( y = e^{x}(x + 1) \) a solution of \(y' = e^{x}y?\)

4. Is \( y = x \arcsin x + \sqrt{1 - x^2} \) a solution of \( \frac{d^2y}{dx^2} = \frac{1}{\sqrt{1 - x^2}}?\)

5. Is \( x^2y + e^{y}x + 7 = 0 \) an (implicit) solution of \((2xy - e^{y})\,dx + (x^2 + e^{y}x)\,dy = 0?\)

6. Is \( x^2 + y^2 + 1 = 0 \) an (implicit) solution of \( y' = \frac{-x}{y}?\)
   (\text{Hint: Careful here. . .})

Creating Differential Equations

We can flip the previous process on it’s head a bit, and create fairly complicated differential equations for which we know a family of solutions. For example, starting with \( y \) defined implicitly as \( xy + \sin y = 0,\)

we can implicitly differentiate and find

\[ y + xy' + y' \cos y = 0, \]

which, by definition, is a differential equation with \( xy + \sin y = 0 \) as an implicit solution. Try your hand at this in the following exercises.

7. Find a first-order differential equation for which \( y^2 + yx + x^2 + 5 = 0 \)

   is an implicit solution.

8. Find a first-order differential equation for which \( e^{y+x} + e^{-y-x} + 1 = 0 \)

   is an implicit solution.

9. Find a second-order differential equation for which \( x^2 + y^2 - 1 = 0 \)

   is an implicit solution.
Direction Fields and Integral Curves

We discussed a nice geometric interpretation of a first-order differential equation in the form
\[ y' = f(x, y), \]
namely that \( f(x, y) \) represents the slope of the solution \( y \) at the point \((x, y)\). Recall the examples we did in class - drawing the direction field for \( y' = \frac{-x}{y} \) lead us to guess circles, and we verified the implicit solution \( x^2 + y^2 = r^2 \). Although we didn’t know all solutions to \( y' = 1 + x - y \), the solution \( y = x \) did appear quickly. In these problems, draw enough isoclines and slope lines until you have a picture of what most solutions would be, graphically, and find one you can represent with an analytic solution. Then, verify your analytic solution does, in fact, solve the ODE. (Be sure the ODE is in proper form first!)

10. \( xy' = -y \).
11. \( y' = 2x + y \).
12. \( (x + y)y' = 1 \).

13. Verify your solutions to the above problems by using a direction field plotter, for instance https://tube.geogebra.org/student/m42741. Try out some other vector fields to get a sense of their solutions. (Suggested ones: \( x^2 - y^2 \), \( x^2 + y \), \( x + y^2 \).) What happens when \( f(x, y) \) is just a function of \( x \), or just a function of \( y \)?

Separation of Variables

We discussed two ways of solving ODEs by separation of variables, the first by getting the ODE into the form
\[ g(x) \, dx + h(y) \, dy = 0 \]
and then integrating, finding an implicit solution
\[ f(x, y) = \int g(x) \, dx + \int h(y) \, dy + C = 0. \]

As an entirely equivalent method*, we discussed putting the differential equation into the form
\[ h(y)y' = g(x) \]
and then integrating both sides with respect to \( x \), yielding
\[ \int h(y)y' \, dx = \int g(x) \, dx + C \]

* In fact, the first method is “short hand” for the second, which we proved in class using \( u \)-substitution.
as an implicit solution. In the following exercises, determine if the equation is separable (i.e. if we can put it in one of the above equivalent forms), and if so find the general solution using the preceding methods.

14. \( x^3 \, dx + y \, dy = 0 \).
15. \( y \, dx + x \, dy = 0 \).
16. \( \sin x \, dx + (y + x) \, dy = 0 \).
17. \( \frac{dr}{d\theta} = -\sin \theta \).
18. \( y' = x + y \).
19. \( y' = 1 + y \).

**Initial Value Problems**

Find the particular solution of the initial value problem given by first finding the general solution and then substituting to determine the value of the constant.

20. \( dx + 3y^2 \, dy = 0 \), \( y(0) = 5 \).
21. \((1 - x) \, dy = x(y + 1) \, dx \), \( y(0) = 0 \).
22. \( y' = \frac{1}{x(1 + 3y^2)} \), \( y(1) = 0 \).

**Challenge Problems**

23. Could \( xy + e^y + 5 = 0 \) be the solution to a separable differential equation?

24. Find a differential equation whose integral curves are circles of radius \( R \), centered at the point \((R, 0)\).

25. Find the general solution of \( yx^2 \, dy - y^3 \, dx = 2x^2 \, dy \).

26. Find the general solution of \( e^{y^2} (x^2 + 2x + 1) \, dx + (xy + y) \, dy = 0 \).

27. Find all continuous functions \( f(x) \) such that \( f(x) = 5 + \int_1^x f(t) \, dt \).

28. A nonnegative function \( f \), continuous on the whole real axis, has the property that its ordinate set over an arbitrary interval has an area proportional to the length of the interval. Find \( f \).