Exam 1 Review

I have shuffled the order of the questions to help you better prepare for the exam. Note that, on the exam, you will be expected to explain your answers and show your work. In particular, if a question has a "yes" or "no" answer, you will not receive any points for simply saying "yes" or "no" if you do not have sufficient work to back up your answer.

A good rule of thumb is to write responses which would serve as adequate explanation to someone who is about 3 weeks behind you - you don’t need to show me how you are multiplying two numbers together or doing a simple integral, but if (for instance) you are asked if an equation is exact you should clearly show where you took partial derivatives to test if it was exact, rather than simply saying “yes” or “no”.

1. Does \( xy + y' = 4x^2 - y^2 \) have a solution for which \( y(1) = 4 \)? If so, is it a unique solution?

2. Find the particular solution of \( y' + 2 = (y + 2x + 1)^{2/3} \) for which \( y(0) = 1 \).

3. An object which is currently 30°C is submerged in a medium whose temperature fluctuates as \( 6 \sin(\pi t) + 30 \), where \( t \) is the time in minutes.
   (a) Setup a differential equation for the temperature \( T \) of the object at a given time \( t \) (in minutes).
   (b) After 1 minute, the temperature of the object is found to be 33°C. Find the corresponding particular solution of the differential equation for \( T(t) \). [You may omit solving for the constants here.]
   (c) What is the maximum temperature of the object?

4. Is \( xy + x^2e^y + 5 = 0 \) an (implicit) solution of \( (ye^{-y} + 2x) \, dx + (xe^{-y} + x^2) \, dy = 0 \)?

5. Determine the set of initial conditions \( (x_0, y_0) \) for which \( (x + y)y' = 2 \) is guaranteed to have a solution. Also determine the set of initial conditions where this solution is guaranteed to be unique.

6. A radioactive isotope decays at a rate proportional to the amount present.
   (a) Setup a differential equation modeling the amount of the isotope \( y \) present at time \( t \).
   (b) If the half-life of the isotope is 28 days, how many days will it take for only 1% of the original amount to be left?

7. Draw a direction field for the ODE \( xy' = y \). Your direction field should include at least isoclines and corresponding slope lines for slopes of value \(-2, -1, 0, 1, 2\). Indicate clearly at least one solution curve. Guess one analytic solution, and show that it is a solution by differentiating.

8. Draw a direction field for the ODE \( y' = x^2 + y^2 \). Your direction field should include at least isoclines and corresponding slope lines for slopes of value \( 0, 1/2, 1, 2, 3 \). Indicate clearly at least one solution curve.
9. Find the particular solution of $(2xy^2 + 2x + 1) \, dy = (-2y^3 - 2y) \, dx$ for which $y(0) = 1$.

10. An object which is 100 °C is placed in a water bath, which is kept at a constant temperature of 10 °C.
    (a) Setup a differential equation for the temperature $T$ of the object at a given time $t$ (in minutes).
    (b) A second measurement is taken, and after 1 minute the temperature of the object has dropped to 90 °C. Using this information, find the corresponding particular solution of the differential equation for temperature $T(t)$.
    (c) At what time will the object reach 20 °C?
    (d) What is the greatest lower bound on the object’s temperature? (i.e. the number $M$ such that $T(t) \geq M$ for all $t$, and if $N > M$, there is a time $t_0$ such that $T(t_0) \leq N$.)

11. Find the general solution of $3x \, dx + 5y \, dy = 0$ using separation of variables.

12. Find the general solution to $y' = -1 - (x + y) \cos x$ using an integrating factor of the form $(x + y)^a$.

13. A tank contains 100 L of water. A salt solution with $1 \, \text{kg/L}$ of salt is added at a rate of $5 \, \text{L/min}$. The solution in the tank is kept uniformly mixed by stirring, and the mixture is pumped out of the tank at a rate of $1 \, \text{L/min}$.
    (a) Setup a differential equation describing the amount of salt $S$ in the tank at $t$ seconds.
    (b) Solve the differential equation to find a function for $S(t)$, the amount of salt in the tank at $t$ seconds.
    (c) At what time will the tank contain 10 kg of salt?
    (d) What is the least upper-bound on the amount of salt in the tank? (i.e. the number $M$ such that $S(t) \leq M$, and if $N < M$ there exists a time $T$ such that $S(T) > N$.)

14. Find the general solution to $\frac{x^2y^2 + y^3 + 1}{x^2y^2 + 1} \, dx + \frac{x}{x^2y^2 + 1} \, dy = 0$.

15. Find the particular solution to $(2 - x) \, dy = x(y^2 + 1) \, dx$ with $y(0) = 1$ using separation of variables.

16. Find the particular solution to $\sqrt{1 - x^2} \, y' = xy$ with $y(0) = 1$ using separation of variables.

17. Is $2 \cos y \, dx + 2x \sin y \, dy = y \sin(xy) \, dx + x \sin(xy) \, dy$ an exact differential equation? Explain your answer.

18. Determine the set of initial conditions $(x_0, y_0)$ for which $y' = \sec(y) + x$ is guaranteed to have a solution. Also determine the set of initial conditions where this solution is guaranteed to be unique.

19. Determine the set of initial conditions for which $yy' = \sin x$ is guaranteed to have a unique solution.
Solve this differential equation (using any method you’d like) to find the general solution. Is there a solution for which \( y(0) = 0 \)? If so, is it unique?

20. Consider the following ODE:
   \[ y' = y^2 x. \]
   (a) Find the general solution
   (b) Is there a solution for which \( y(0) = 0 \)?
      i. If so, write down the solution. Does the uniqueness theorem guarantee this is the only solution? (Explain your answer.)
      ii. If not, explain why this does not contradict the existence theorem.

21. Consider the following ODE:
   \[ y' = y^2 \cos x. \]
   (a) Find the general solution.
   (b) Find the particular solution for which \( y(\pi/2) = 0 \).
   (c) Is your particular solution with \( y(\pi/2) = 0 \) guaranteed to be the only one (by the Uniqueness Theorem)?

22. Is \( y = \sin x \) a solution to \( y' + y = 0 \)? What about \( y'' + y = 0 \)?

23. Consider the following ODE:
   \[ (x + 2y)y' = -y + 2x. \]
   (a) Find the general solution.
   (b) Is there a solution for which \( y(2) = -1 \)?
      i. If so, write down the solution. Does the uniqueness theorem guarantee this is the only solution? (Explain your answer.)
      ii. If not, explain why this does not contradict the existence theorem.

24. Consider the following ODE:
   \[ y' = \frac{3x^2 - y}{x}. \]
   (a) Find the general solution.
   (b) Find the particular solution for which \( y(0) = 0 \).
   (c) Why is the existence of a particular solution with \( y(0) = 0 \) not a contradiction of the Existence Theorem?

25. A tank containing 100 L of water has 10 kg of salt dissolved in it. A plug at the bottom of the tank is opened, allowing a uniform mixture from the tank to flow out at a rate of \( 1 \text{ L/min} \), as fresh water is added at an equal rate.
(a) Setup a differential equation describing the amount of salt \( S \) in the tank at \( t \) minutes.
(b) Solve the differential equation to find a function for \( S(t) \), the amount of salt in the tank at \( t \) seconds.
(c) At what time will half the salt be gone?
(d) At what time will all the salt be gone?

26. Is \( x^2 y \, dx + (3y + x) \, dy = 0 \) an exact differential equation? Explain your answer.

27. Find the particular solution of \( 2y^2 \, dx + (1 + 2xy) \, dy = 0 \) for which \( y(2) = 1 \).

28. Is \( 2xy \, dx + (x^2 + y^2) \, dy = 0 \) an exact differential equation? Explain your answer.

29. Find a second-order differential equation for which \( x^2 + 1 = y^2 \) is an implicit solution.

30. The rate of growth of a particular bacteria is proportional to the current amount of bacteria.
   (a) Setup a differential equation modeling the amount of bacteria \( B \) present at time \( t \) (in days).
   (b) At time \( t = 0 \), there are 500 bacterial cells present. After 1 day the amount of bacteria has tripled
to 1500. Using this new information, find the corresponding particular solution for the amount of
   bacteria \( B(t) \).
   (c) When will the amount of bacteria reach 5000 cells?

31. Find the general solution to \( y' = y + e^x \).

32. Find a first-order differential equation for which \( y^3 + xy + x + 1 = 0 \) is an implicit solution.

33. A 10 kg object is dropped from a hot air balloon. It hits the ground 1 minute later. Assume the force of
   air resistance is \(-10v\).
   (a) Setup an initial value problem for the velocity \( v \) at time \( t \), where \( t \) is in seconds.
   (b) Solve the initial value problem for the velocity \( v(t) \).
   (c) How high was the air balloon when the object was dropped?

34. Find the general solution to \( y = (x + \frac{x^2}{y^2+1})y' \).

35. Is \( 2x \, dx + 2y \, dy = 0 \) an exact differential equation? Explain your answer.

36. Find the general solution to \( yy' + y^2 + 1 = x \sqrt{y^2 + 1} \) by making a substitution \( u = \sqrt{y^2 + 1} \).

37. Find the general solution to \( (y^2 + x)(2yy' + 1) = 1 \). (Hint: Use a substitution.)
38. Is \( y = xe^x \) a solution to \( y'' + y = x? \)

39. Find the particular solution to \( y' = 3x + xy \) with \( y(0) = 1 \) using separation of variables.

40. Find a first-order differential equation for which \( \sin^2 x + \cos^2 y = 1 \) is an implicit solution.

41. Find the general solution to \( \cos(x + y) \, dx + \cos(x + y) \, dy = 0. \)

42. Draw a direction field for the ODE \( y' = x + 3y \). Your direction field should include at least isoclines and corresponding slope lines for slopes of value \(-2, -1, 0, 1, \) and \(2\). Indicate clearly at least one solution curve. Guess one analytic solution, and show that it is a solution by differentiating.

43. Find the general solution to \( (y + 2xy \cos^2(xy)) \, dx + (x + x^2 \cos^2(xy)) \, dy = 0 \) using the integrating factor \( \sec^2(xy) \).

44. A population of rabbits reproduce at a rate proportional to the total population.
   (a) Setup a differential equation modeling the number of rabbits \( R \) at a given time \( t \) (in days).
   (b) Suppose the total rabbit population doubles in 50 days. Approximately how many rabbits should you start with if you needed 400 rabbits in 180 days?

45. Show that there is no particular solution of \( y' + \frac{1}{x}y = 4x^2 \) with \( y(0) = 1. \)

46. Find the general solution of \( y' = -\frac{x}{y} \) using separation of variables.

47. A 5 kg object is thrown straight up with a velocity of \( 30 \text{ m/s} \). Assume the force of air resistance is \(-5v\).
   (a) Setup an initial value problem for the velocity \( v \) at time \( t \), where \( t \) is in seconds.
   (b) Solve the initial value problem to find the velocity \( v(t) \).
   (c) At what time does the ball reach maximum height? (You do not need to solve for position, \( r(t) \), in order to find this.)

48. An object which is \( 60 \degree \text{C} \) is placed in a water bath, whose temperature is controlled as \( 3 \sin(\pi t) + 10 \degree \text{C} \), with \( t \) representing minutes. After 1 minute, the temperature of the object is \( 40 \degree \text{C} \).
   (a) Setup an initial value problem describing the temperature \( T \) in the tank at \( t \) minutes.
   (b) Solve the differential equation to find a function for \( T(t) \). [Don’t worry about solving for the constants here.]
   (c) What is the temperature of the object after 2 minutes? [You may omit this.]
   (d) Will the temperature of the object ever reach 10 \degree \text{C}? [You may omit this.]
49. Find the particular solution to $y' + \frac{1}{2} y = y^2$ with $y(1) = 1$ by using an integrating factor $-y^{-2}$ and making a substitution $u = y^{-1}$.

50. A tank containing 200 L of fresh water begins receiving salt water of an unknown concentration at a rate of 1 L/min. The solution is uniformly mixed and it flows out at the same rate through a hole in the bottom. After 1 minute, there are 10 kg of salt in the tank.

(a) Setup an initial value problem describing the amount of salt $S$ in the tank at $t$ minutes.
(b) Solve the differential equation to find a function for $S(t)$, the amount of salt in the tank at $t$ minutes.
(c) What is the concentration of the incoming salt water?
(d) What is the least upper bound for the amount of salt in the tank? (i.e. a number $M$ such that $S(t) \leq M$, and if $N < M$ there is some time $T$ such that $S(T) > N$.)