1. \[ xy + y' = 4x^2 - y^2 \Rightarrow y' = 4x^2 - y^2 - xy. \]

The solution is continuous because \( 4x^2 - y^2 - xy \) is continuous everywhere, in particular near \((1,4)\), so a solution exists. Moreover, \[
\frac{\partial}{\partial y}(4x^2 - y^2 - xy) = -2y - x \]

is continuous near \((1,4)\), so this solution is unique.

2. \[ y' + 2 = (y + 2x + 1)^{2/3} \]

is not separable or linear, and the presence of the \(2/3\) exponent makes it unlikely to be exact. However, if we let \( u = y + 2x + 1 \), \[ u' = y' + 2, \] so \[ u' = u^{2/3} \]

\[ \int u^{-3/2} du = \int dx \]

\[ 3u^{1/3} = x + C \]

\[ (y + 2x + 1)^{\frac{1}{3}} = \frac{x}{3} + C \]

\[ y + 2x + 1 = (\frac{x}{3} + C)^3 \]

\[ y = (\frac{x}{3} + C)^3 - 2x - 1 \]

3. a) Let\( T(t) = \) TEMP OF BODY AT TIME \( t \), \( M(t) = \) TEMP OF MEDIUM, \( t = \) TIME in MINUTES,

\[
\begin{cases}
T'(t) = -k(T(t) - M(t)) = -k(T(t) - (6\sin(\pi t) + 30)) \\
T(0) = 30
\end{cases}
\]

b) \( T' + kT = k(6\sin(\pi t) + 30) \)

is linear, so

\[ T(t) = e^{-kt} \int e^{kt} (6\sin(\pi t) + 30) \, dt \]

\[ = e^{-kt} \left[ 6k \int e^{kt} \sin(\pi t) \, dt + 30k \int e^{kt} \, dt \right] \]

Now \[ \int e^{kt} \sin(\pi t) \, dt \]

let \( u = \sin(\pi t) \) \[ \Rightarrow du = \pi \cos(\pi t) \, dt \]

\[ v = \frac{k}{\pi} e^{kt} \]

Let \( w = \cos(\pi t) \)

\[ dw = -\pi \sin(\pi t) \, dt \]

\[ = \frac{e^{kt}}{k} \sin(\pi t) - \frac{\pi}{k} \left[ \frac{e^{kt}}{k} \cos(\pi t) + \frac{k}{\pi} \int e^{kt} \sin(\pi t) \, dt + C \right] \]
\[ \int e^{kt} \sin(mt) \, dt = \frac{e^{kt}}{K} \sin(mt) - \frac{me^{kt}}{K^2} \cos(mt) - \frac{m^2}{K} \int e^{kt} \sin(mt) \, dt + C \]

\[ (1 + \frac{m^2}{K^2}) \int e^{kt} \sin(mt) \, dt = \frac{e^{kt}}{K} \left( \sin(mt) - \frac{m}{K} \cos(mt) \right) + C \]

\[ \int e^{kt} \sin(mt) \, dt = \frac{e^{kt}}{K(1 + \frac{m^2}{K^2})} \left( \sin(mt) - \frac{m}{K} \cos(mt) \right) + C. \]

Also, \( S e^{kt} \, dt = \frac{1}{K} e^{kt} + C \), so

\[ T(t) = e^{-kt} \left[ \frac{6e^{kt}}{1 + \frac{m^2}{K^2}} \left( \sin(mt) - \frac{m}{K} \cos(mt) \right) + 30e^{kt} + C \right] \]

We have two conditions to apply in order to find \( K \) and \( C \):

\[ T(0) = 30, \quad T(1) = 33. \]

\[ T(0) = \frac{6}{1 + \frac{m^2}{K^2}} \left( - \frac{m}{K} \right) + 30 + C = 30 \]

\[ \Rightarrow C = \frac{6mK}{K^2 + m^2} \]

\[ T(1) = e^{-k} \left[ \frac{6e^{kt}}{1 + \frac{m^2}{K^2}} \left( \sin(mt) - \frac{m}{K} \cos(mt) \right) + 30e^{kt} + C \right] = 33 \]

\[ \Rightarrow Ce^{-k} = 3 - \frac{6mK}{K^2 + m^2} = 3 - C \]

\[ \Rightarrow C = \frac{3}{1 + e^{-k}} \]

Solving this system:

\[ \begin{cases} C = \frac{6mK}{K^2 + m^2} \\ C = \frac{3}{1 + e^{-k}} \end{cases} \]

Can be done numerically (by computer), but on an exam (where you cannot use a calculator) I would give you \( K \) and \( C \), or they would be easier to compute.

// You may omit solving for \( K \) and \( C \) here.
(4) There are many ways to check this, here's one:
\[
\frac{\partial}{\partial x}(xy + x^2e^y + 5) = y + 2xe^y
\]
\[
\frac{\partial}{\partial y}(xy + x^2e^y + 5) = x + x^2e^y
\]
So
\[
y' = \frac{y + 2xe^y}{x + x^2e^y}
\]
\[
\Rightarrow (y + 2xe^y)dx + (x + x^2e^y)dy = 0
\]
\[
\Rightarrow (ye^{-y} + 2x)dx + (xe^{-y} + x^2)dy = 0
\]
(by mult. by \(e^{-y}\)).
Thus it is a solution.

(5) \((x+y)y' = 2 \Rightarrow y' = \frac{2}{x+y} \quad (if \ y \neq x)\),
this is continuous at all \(x \neq y\) and therefore the existence theorem.
\[\Rightarrow \exists \text{ a solution with } y(x_0) = y_0 \quad \forall x_0 \neq y_0.\]
Also, \[\frac{d}{dy} \left( \frac{2}{x+y} \right) = \left(\frac{2}{x+y}\right)^2 , \text{ and hence as long as } x_0 \neq y_0,\]
\[\Rightarrow \text{ the solution is unique.}\]

(6) \(y' = -ky, \ y = \text{amount of isotope present}, \ t = \text{time (in days)}\)
(6) The half-life is the amount of time for half this

original amount to decay. Solving:
\[
s\frac{dy}{dt} = \{-k\}
\Rightarrow \ln|y| = -kt + C
\Rightarrow y(0) = Ce^{-kt}, \text{ and } y(0) = C
\Rightarrow y(28) = \frac{1}{2} y(0) = \frac{1}{2} C
\Rightarrow \frac{1}{2} C = Ce^{-k28} \Rightarrow (\frac{1}{2})^{\frac{1}{28}} = e^{-k}, \text{ hence}
\Rightarrow y(t) = C (\frac{1}{2})^{t/28}
\]
Now we seek \(T\) where \(y(T) = \frac{C}{100} \Rightarrow C (\frac{1}{2})^{T/28} = \frac{C}{100}\)
\Rightarrow \frac{1}{100} = ln (\frac{1}{2}) T/28 = ln \frac{1}{100} \Rightarrow T = 28 \ln \frac{100}{2}
1. \( xy' = y \Rightarrow y' = \frac{y}{x} \)

\[
\begin{array}{c|c}
\text{Slopes} & \text{Isocline} \\
\hline
-2 & -2x = y \\
-1 & -x = y \\
0 & 0 = y \\
1 & x = y \\
2 & 2x = y \\
\end{array}
\]

\[\text{In this case, these isoclines are solution curves. (This is not always the case!!!)}\]

Solution curve: \( y = 2x \)

Verification: \( y' = 2, \quad xy' = x \cdot 2 = y \) \( \checkmark \), so it is a solution.

2. \( y' = x^2 + y^2 \)

\[\begin{array}{c|c}
\text{Slopes} & \text{Isocline} \\
\hline
-2 & \text{N/A} \\
-1 & \text{N/A} \\
0 & \text{N/A} \\
1 & x^2 + y^2 = 1 \\
2 & x^2 + y^2 = 4 \\
\end{array}\]

Solution curves: \( 3 \) circles

3. \((2xy^2 + 2x + 1)dx + (2y^2 - 2y)dy = 0\), \( y(0) = 1 \)

\[
\Rightarrow (2y^2 + 2y)d\frac{y}{x} + (2xy^2 + 2x + 1)dy = 0
\]

\[
\frac{d}{dx} \left( \frac{y}{x} \right) = \frac{2y^2 + 2y}{2xy^2 + 2x + 1}
\]

\( 6y^2 + 2 = 2y^2 + 2 \)

So it is not exact. We'll look for an integrating factor. The form of the test above indicates we should try \( h = h(y) : \)

\[
h(y)(2y^2 + 2y)dx + h(y)(2xy^2 + 2x + 1)dy = 0
\]

\[
\frac{d}{dy} \left( \frac{y}{x} \right) = \frac{h(y)(6y^2 + 2) + h(y)(6y^2 + 2)}{h(y)} = h(y)(6y^2 + 2)
\]

\[\Rightarrow \frac{h'}{h} = -4y^2 \Rightarrow -2y \]
\[ \ln h = \int \frac{-2y}{y^2+1} \, dy = -\int \frac{2y}{y^2+1} \, dy = -\ln(y^2+1) \]
\[ \Rightarrow h(y) = \frac{1}{y^2+1}, \text{ so we find the exact equation} \]
\[ \frac{2y^3+2y}{y^2+1} \, dx + \frac{2xy^2+2x+1}{y^2+1} \, dy = 0 \]
\[ \Rightarrow 2y \, dx + (2x + \frac{1}{y^2+1}) \, dy = 0 \]

Thus
\[ f(x, y) = \int 2y \, dx = 2xy + C(y) \]
\[ f_y(x, y) = 2x + C'(y) = 2x + \frac{y}{y^2+1} \Rightarrow C'(y) = \frac{y}{y^2+1} \]
\[ \Rightarrow C(y) = \int \frac{1}{y^2+1} \, dy = \arctan y + C \]
\[ \Rightarrow f(x, y) = 2xy + \arctan y + C = 0 \text{ is an implicit solution.} \]

Now we substitute:
\[ 2(0)(1) + \arctan 1 + C = 0 \Rightarrow \frac{\pi}{4} + C = 0 \Rightarrow C = -\frac{\pi}{4} \]

So
\[ 2xy + \arctan y - \frac{\pi}{4} = 0 \text{ is the particular solution,} \]

\[ \text{at } T(t) = -k(T(t)-M) = -k(T(t)-10) \]
\[ \Rightarrow T(t) = 100 \]

where \( T \) is the temp of the object, and \( t \) is time in minutes.

\[ T'(1) = 90 \]
\[ T' = -k(T-10) \Rightarrow \int \frac{dT}{T-10} = \int -k \, dt \]
\[ \Rightarrow \ln |T-10| = -kt + C \Rightarrow T(t) = Ce^{-kt} + 10, \]
\[ T(0) = 100 \Rightarrow C + 10 = 100 \Rightarrow C = 90, \]
\[ T(1) = 90 \Rightarrow 90e^{-k} + 10 = 90 \Rightarrow e^{-k} = \frac{8}{9}, \text{ so} \]
\[ T(t) = 90 \left( \frac{8}{9} \right)^t + 10 \text{ is the particular solution.} \]

\[ T(t_0) = 20 \Rightarrow 90 \left( \frac{8}{9} \right)^{t_0} + 10 = 20 \]
\[ \Rightarrow \left( \frac{8}{9} \right)^{t_0} = \frac{1}{9} \Rightarrow (\ln \frac{8}{9}) t_0 = \ln \left( \frac{1}{9} \right) \Rightarrow t_0 = \frac{\ln \frac{8}{9}}{\ln \frac{8}{9}}, \]

We're looking for a minimum.
\[ T'(t) = \ln \left( \frac{8}{9} \right) 90 \left( \frac{8}{9} \right)^t \neq 0, \text{ so we check the "endpoints":} \]
\[ T(8) = 100, \text{ and } \lim_{t \to 8^+} T(t) = 10, \text{ thus the greatest lower bound is } 10°C. \]
(11) \[ 3x \, dx + 5y \, dy = 0 \]
\[ \Rightarrow \int 5y \, dy = -\int 3x \, dx \Rightarrow \frac{5}{2} y^2 = -\frac{3}{2} x^2 + C \]
\[ \Rightarrow 5y^2 + 3x^2 + C = 0. \]

(12) \[ y' = -1 - (x+y) \cos x \]
\[ \Rightarrow (1 + (x+y) \cos x) \, dx + dy = 0 \]
\[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) \cos x = 0 \]

// Even though it suggests an integrating factor of the form \((x+y)^a\), this is screaming for \( h = h(x) \), so I'll do both and compare results.
\[ h(x) \left( (x+y) \cos x \right) \, dx + h(x) \, dy = 0 \]
\[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) \cos x = h'(x) \]
\[ \Rightarrow \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) \cos x = \frac{h(x)}{h(x)} \cos x \Rightarrow \ln h = \int \cos x \, dx = \sin x, \]
So \( h(x) = e^{\sin x} \). // Ugh, maybe that's why I suggested \((x+y)^a\).

Now we solve \( e^{\sin x} \left( (1 + (x+y) \cos x) \, dx + e^{\sin x} \, dy = 0 \right. \)
\[ f(x,y) = \int e^{\sin x} \, dy = ye^{\sin x} + c(x) \]
\[ f(x,y) = (x+y)e^{\sin x} + c'(x) = e^{\sin x} \left( (1 + (x+y) \cos x) \right) \]
\[ \Rightarrow c'(x) = e^{\sin x} + x \cos x \, e^{\sin x} \]

// Yuck; this doesn't look too pleasant to integrate.

Maybe I'll revert to my earlier suggestion:
\[ h(x,y) = (x+y)^a \Rightarrow (x+y)^a \left( (1 + (x+y) \cos x) \, dx + (x+y)^a \, dy = 0 \right. \]
\[ \Rightarrow ((x+y)^a + (x+y)^{a+1} \cos x) \, dx + (x+y)^a \, dy = 0 \]
\[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) \cos x = a(x+y)^{a-1} + (a+1)(x+y)^a \cos x = a(x+y)^{a-1} \]
\[ \Rightarrow (a+1)(x+y)^a \cos x = 0 \]
\[ \Rightarrow x = y \text{ or } a = -1. \]

Note that \( y(x) = x \Rightarrow y'(x) = 1 \), and \( 1 \neq -1 \) so \( y(x) : x \) does not solve the ODE, and thus we must have \( a = -1 \).
\[ (\frac{1}{x+y} + \cos x)\,dx + (\frac{1}{x+y})\,dy = 0 \]

Now \( f(x, y) = \int \frac{1}{x+y} + \cos x \,dx = \ln|x+y| + \sin x + C_1(y) \),

and \( f(x, y) = \int \frac{dy}{x+y} = \ln|x+y| + C_2(x) \)

So, taking \( C_1(y) = C \), \( C_2(x) = \sin x + C \), we find \( \ln|x+y| + \sin x + C = 0 \) is an implicit solution.

We can convert this to an explicit solution:

\[ |x+y| = e^{-\sin x - C} \rightarrow x+y = Ce^{-\sin x} \]

\[ \Rightarrow y(x) = Ce^{-\sin x} - x \]

/* Our previous analysis using \( h(x) = e^{\sin x} \) was not wrong, it just lead to a more difficult integral. This demonstrates that the integrating factor is not unique, and some choices may lead to an 'easier' exact equation than another.

As an aside, we could actually use our result here to solve \( \int e^{\sin x} + x\cos x e^{\sin x} \,dx \):

From before, we had \( f(x, y) = ye^{\sin x} + c(x) \), and \( f(x, y) = 0 \) would be our solution, so \( y(x) = -e^{\sin x} \cdot c(x) \).

On the other hand, we've now found

\[ y(x) = Ce^{\sin x} - x = e^{-\sin x} (C + xe^{\sin x}) \], so \( c(x) = xe^{\sin x} + C \), thus \( \int e^{\sin x} + x\cos x e^{\sin x} \,dx = xe^{\sin x} + C \), as you may verify. */

13. @ \( S = \text{Amount of Salt (in lbs)} \)
\[ t = \text{Time (in seconds)} \]
\[ S'(t) = 5 - \frac{5}{W(t)} \cdot 1 \]
\[ S(0) = 0 \]

where \( W(t) \) is the amount of water at time \( t \).
\[ W'(t) = 5 - 1 = 4 \Rightarrow W(t) = 4t + C, \]
\[ W(0) = 100 \]
\[ \Rightarrow W(t) = 4t + 100, \text{ and so} \]
\[
\begin{align*}
S' &= 5 - \frac{S}{4t + 100} \\
S(0) &= 0
\end{align*}
\]

\[S'(t) + \frac{S}{4t + 100} = 5 \quad \text{is separable, so}
\]

\[h(t) = e^\int \frac{dt}{4t + 100} = e^{\frac{1}{4} \ln (4t+100)} = (4t+100)^{\frac{1}{4}}
\]

AND \[
S(t) = (t+25)^\frac{1}{4} \left[ (t+25)^\frac{3}{4} \cdot 5 \right] dt
\]

\[= (t+25)^\frac{1}{4} \left[ 5 \frac{1}{3} (t+25)^{\frac{3}{4}} + C \right]
\]

\[= 4t+100 + \frac{C}{(t+25)^{\frac{1}{4}}}
\]

\[\text{Hence } S(0) = 0 \Rightarrow 100 + \frac{C}{25^{\frac{1}{4}}} = 0 \Rightarrow C = -100 \cdot 25^{\frac{1}{4}},
\]

AND SO THIS PARTICULAR SOLUTION IS \[
S(t) = 4t+100 + \frac{-100 \cdot 25^{\frac{1}{4}}}{(t+25)^{\frac{1}{4}}}
\]

\[S(t) = 10 \Rightarrow 4t + 100 - 100 \cdot 25^{\frac{1}{4}} / (t+25)^{\frac{1}{4}} = 10
\]

\[\Rightarrow (4t+90)^{\frac{1}{4}} = 100^{\frac{1}{4}} \cdot 25^{\frac{1}{4}} / (t+25)
\]

\[\Rightarrow 2500000000000000000000000000000000000
\]

This is a 5th degree polynomial, which we can solve with a computer to find \[T \approx 2.01923\]

//On this exam, the highest degree polynomial I would ask you to solve would be degree 2, so you could use the quadratic formula.

\[S'(t) = 4 - 100 \cdot \frac{1}{4} \left( -\frac{1}{4} \right) (t+25)^{-\frac{5}{4}}, \text{so for } t > 0 \]

\[S'(t) > 0 \text{ which means it is always increasing, thus our least upper bound will be }
\]

\[\lim_{t \to \infty} S(t) = (\infty),
\]

ie. There is NO upper bound!
\[ 14 \quad \frac{x^2 y^2 + y + 1}{x^2 y^2 + 1} \, dx + \frac{x}{x^2 y^2 + 1} \, dy = 0 \]

\[ \Rightarrow (1 + \frac{y}{x^2 y^2 + 1}) \, dx + \frac{x}{x^2 y^2 + 1} \, dy = 0 \]

\[ \frac{2}{\frac{\partial y}{\partial x}} \left( \frac{x^2 y^2 + 1 - y(2xy)}{(x^2 y^2 + 1)^2} \right) = \frac{2}{\frac{\partial x}{\partial x}} \left( \frac{x^2 y^2 + 1 - x(2xy)}{(x^2 y^2 + 1)^2} \right) \]

\[ \frac{-x^2 y^2 + 1}{(x^2 y^2 + 1)^2} = \frac{1}{x^2 y^2 + 1} \]

So it is exact.

\[ \int 1 + \frac{y}{x^2 y^2 + 1} \, dx = x + \int \frac{y}{(x y^2 + 1)} \, dx = x + \arctan(xy) + c_1(y) \]

\[ \int \frac{x}{x^2 y^2 + 1} \, dy = \arctan(xy) + c_2(x) \]

So, taking \( c_2(x) = x + C, \ c_1(x) = C \), we find

\[ x + \arctan(xy) + C = 0 \]

is an implicit solution.

We can (and should) make this explicit:

\[ \arctan(xy) = C - x \]

\[ \Rightarrow xy = \tan(C - x) \Rightarrow y(x) = \frac{1}{x} \tan(C - x) \]

\[ 15 \quad (2-x) \, dy = x(y^2 + 1) \, dx, \ y(0) = 1 \]

\[ \Rightarrow \int \frac{dy}{y^2 + 1} = \int \frac{x}{2-x} \, dx \Rightarrow \arctan y = - \int \frac{x - 2 + 2}{x - 2} \, dx \]

\[ = - \int (1 + \frac{2}{x - 2}) \, dx = -x - 2 \ln|x - 2| + C \]

Thus \( y(x) = \tan(-x - 2 \ln|x - 2| + C) \).

\[ y(0) = \tan(-2 \ln 1 - 2 | + C) = 1 \Rightarrow -2 \ln 1 - 2 | + C = \frac{\pi}{4} \]

\[ \Rightarrow C = \frac{\pi}{4} + 2 \ln 2, \text{ thus } y(x) = \tan(-x - 2 \ln|x - 2| + \frac{\pi}{4} + 2 \ln 2) \]
\[ \sqrt{1-x^2} \quad y' = xy \quad \text{with} \quad y(0) = 1 \]

\[ \frac{dy}{y} = \int \frac{x}{\sqrt{1-x^2}} \, dx \quad \Rightarrow \quad \ln|y| = -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = -\frac{1}{2} 2\sqrt{u} + C \]

\[ \Rightarrow \ln|y| = -\sqrt{1-x^2} + C \quad \Rightarrow \quad y(x) = Ce^{-\sqrt{1-x^2}} \]

\[ \text{Now} \quad y(0) = 1 \quad \Rightarrow \quad 1 = Ce^{-\sqrt{1}} \quad \Rightarrow \quad C = e \quad \text{so} \quad y(x) = e^{-\sqrt{1-x^2}}. \]

\[ 2\cos y \, dx + 2x \sin y \, dy = y \sin(xy) \, dx + x \sin(xy) \, dy \]

\[ \Rightarrow \quad (2\cos y - y \sin(xy)) \, dx + (2x \sin y - x \sin(xy)) \, dy = 0 \]

\[ \frac{\partial}{\partial y} (-2 \sin y - \sin(xy)) - x \cos(xy) = 2 \sin y - \sin(xy) - x \cos(xy) \]

This equation is not exact, because \( P_y \neq Q_x \) as \( x = y \) and \( Q_x \) are continuous, but not equal, as shown above. Thus, no function \( f(x,y) \) s.t. \( f_x = P \) and \( f_y = Q \) could exist.

\[ y' = \sec y + x \quad \text{is guaranteed to have a solution through a point} \quad (x_0, y_0) \quad \text{whenever} \quad \sec y + x \quad \text{is continuous near} \quad (x_0, y_0) \quad \text{which is the case as long as} \quad y_0 \neq \frac{\pi}{2} + \pi n \quad \text{for an} \quad n \in \mathbb{Z}. \quad \text{(integers)} \]

The solution is guaranteed to be unique as long as

\[ \frac{\partial}{\partial y} (\sec y + x) = -\tan y \sec y \]

is continuous near \( (x_0, y_0) \), which is again the case as long as \( y_0 \neq \frac{\pi}{2} + \pi n \) for an \( n \in \mathbb{Z} \).

\[ y y' = \sin x \quad \Rightarrow \quad y' = \frac{\sin x}{y}, \quad y \neq 0. \]

This is guaranteed to have a solution through \( (x_0, y_0) \) as long as \( \frac{\sin x}{y} \) is continuous near \( (x_0, y_0) \), which is the case as long as \( y_0 \neq 0. \)
The solution is further guaranteed to be unique as long as \( \frac{\partial}{\partial y} \left( \frac{\sin x}{y} \right) = -\frac{\sin x}{y^2} \)

is continuous near \((x_0, y_0)\), which is again the case as long as \(y_0 \neq 0\).

Now, \(yy' = \sin x \implies \int y \, dy = \int \sin x \, dx \)

\(\implies \frac{1}{2}y^2 = -\cos x + C,\)

If \(y(0) = 0, \implies \frac{1}{2} (0) = -\cos (0) + C \implies C = 1,\) so \(\frac{1}{2}y^2 = -\cos x + 1\) would be an implicit solution.

This problem asks us to find one particular solution: \(y(x) = -\frac{\sqrt{2-2 \cos x}}{2}\) works, let's check:

\[ y'(x) = \frac{-\sin x}{\sqrt{2-2 \cos x}} \]

\[ = \frac{-\sin x}{y} \]

So a solution exists on the other hand.

\(y(x) = -\sqrt{2-2 \cos x}\)

is also a solution:

\[ y'(x) = \frac{-\sin x}{\sqrt{2-2 \cos x}} = \frac{-\sin x}{y} \]

So the solution is not unique.

(2a) \(y' = y^2 x\)

(2b) \(y^2 y' = x \) (if \(y \neq 0\))

\[ \implies \int y^2 \, dy = \int x \, dx \implies -\frac{1}{3} = \frac{x^2}{2} + C \]

\[ \implies y(x) = -\frac{x^2}{2} + C \]

We also need to check \(y(x) = 0 \implies y'(x) = 0,\) so \(y' = y^2 x \)

So \(y(x) = 0\) is a solution which was not included in \(y(x) = -\frac{x^2}{2} + C\) ! (This is important!)

(3a) Yes, \(y(x) = 0\).

Considering \(\frac{\partial}{\partial y} (y^2 x) = 2yx,\) which is continuous everywhere, the uniqueness theorem \(\Rightarrow y(x) = 0\) is the only solution with \(y(0) = 0\).
This point of this question was to show how the existence theorem can be useful, and show you solutions you may have missed. If you didn't remember to check $y(x) = 0$ in part (a), you would have had to contend with (c)(ii), which would lead to a contradiction.

2. $y' = y^2 \cos x$
   a. If $y \neq 0$, $y^{-2} y' = \cos x$
      $\Rightarrow \int y^{-2} \, dy = \int \cos x \, dx \Rightarrow -y^{-1} = \sin x + C$

      $\Rightarrow y(x) = \frac{-1}{\sin x + C}$
   
   If $y(x) = 0 \Rightarrow y'(x) = 0$, so $y' = y^2 \cos x$ 
   
   Thus $y(x) = 0$ is a solution which was not included above.

   b. $y(x) = 0$ is a solution s.t. $y(0) = 0$.
   
   c. Inserting $\frac{d}{dx} y^2 \cos x = 2y \cos x$, which is continuous everywhere, we see that $y(x) = 0$ is the only solution by the uniqueness theorem.

22. $y(x) = \sin x \Rightarrow y'(x) = \cos x$, and $y''(x) + y(x) = \cos x + \sin x \neq 0$, so it does not solve $y'' + y = 0$. On the other hand, $y''(x) = -\sin x$, and so $y'' + y = -\sin x + \sin x = 0$, so it does solve $y'' + y = 0$.

23. (a) $(x + 2y) y' = -y + 2x$
   $\Rightarrow (y - 2x) \, dx + (x + 2y) \, dy = 0$

   $\frac{\partial}{\partial y} \left( \frac{y}{x} \right) = \frac{1}{x}$

   So this is exact.

   $\int y - 2x \, dx = xy - x^2 + c_1(y)$

   $\int x + 2y \, dy = xy - y^2 + c_2(x)$

   Taking $c_1(y) = -y^2 + C$, $c_2(x) = -x^2 - C$
We find an implicit solution
\[ xy - x^2 - y^2 + C = 0 \]
\[ y(2) = -1 \implies -1(2) - (-1)^2 - (2)^2 + C = 0 \]
\[ \implies -2 - 1 - 4 + C = 0 \implies C = 7 \]
So \[ xy - x^2 - y^2 + 7 = 0 \] is an implicit solution for which \( y(2) = -1 \). Returning to the original equation, if \[ 2y \neq -x \] we have
\[ y' = \frac{-y + 2x}{x + 2y} \]
Since the right hand side is not continuous at \((2, -1)\), this existence theorem doesn't apply, and so neither does the uniqueness theorem.

BONUS: There are actually two explicit solutions with \( y(2) = -1 \), can you write them down?

24) \[ y' = \frac{3x^2 - y}{x} \]
\[ (y - 3x^2)\,dx + x\,dy = 0 \]
\[ \frac{\partial}{\partial y} y' = \frac{\partial}{\partial y} \frac{3x^2 - y}{x} \]
\[ = 3 \frac{x^2}{x} - \frac{y}{x} \]
\[ = 3x - \frac{y}{x} \]
So this is exact.

\[ f(x, y) = \int x\,dy = xy + C(x), \]
\[ f_x(x, y) = y + c'(x) = y - 3x^2 \implies c'(x) = -3x^2 \]
\[ \implies c(x) = -x^3 + C \]
\[ xy - x^3 + C = 0 \] is an implicit solution, and
\[ y(0) = 0 \implies C = 0, \quad y(x) = x^2 \]

Check: \( y'(x) = 2x \), \[ 2x = \frac{3x^2 - y}{x} \checkmark \]

The existence theorem tells us when a solution is guaranteed to exist, it does not say that anything about a solution not existing. Therefore, despite the fact that \( \frac{3x^2 - y}{x} \) is not continuous at \((0, 0)\),
The fact that we've found a solution \( y(x) = x^2 \) with \( y(0) = 0 \) is not a contradiction of the existence theorem.

1. \( S = \text{amount of salt in kg} \)
   \( t = \text{time in minutes} \)
   \( S' = \text{rate in} - \text{rate out} \)
   \[ S' = 0 - \frac{S}{100} = -\frac{S}{100} \]
   So \( \begin{cases} S' = -\frac{S}{100} \\ S(0) = 10 \end{cases} \)

2. \[ \frac{dS}{S} = -\frac{dt}{100} \Rightarrow \ln |S| = -\frac{t}{100} + C \]
   \[ \Rightarrow S(t) = Ce^{-\frac{t}{100}}, \quad S(0) = 10 \Rightarrow C = 10 \]
   So \( S(t) = 10e^{-\frac{t}{100}} \)

3. \[ S(T) = 5 \Rightarrow 10e^{-\frac{T}{100}} = 5 \Rightarrow e^{-\frac{T}{100}} = \frac{1}{2} \]
   \[ \Rightarrow T = -100 \ln \left( \frac{1}{2} \right) = 100 \ln 2 \text{ minutes} \]

4. \[ S(T) = 0 \Rightarrow 10e^{-\frac{T}{100}} = 0 \neq (\text{contradiction}) \]
   Thus, there is no value of \( T \) for which \( S(T) = 0 \), i.e.,
   this salt is never all gone!
26. \( x^2 y \, dx + (3y+x) \, dy = 0 \)

\[
\begin{align*}
\frac{\partial}{\partial y} \left( x^2 y \right) &= x^2 = \frac{\partial}{\partial x} \left( x^2 y \right) \\
\end{align*}
\]

Since \( x^2 \) and 1 are continuous but not equal, this equation is not exact, since, if it were, we would have \( f_x(x,y) = x^2 y \) and \( f_y(x,y) = 1 \)

\[
\Rightarrow \frac{\partial}{\partial y} f(x,y) = x^2 = \frac{\partial}{\partial x} f(x,y) = 1, \text{ which is false.}
\]

27. \( \frac{d}{dx} \left( x^2 + 1 \right) = \frac{d}{dx} y^2 \Rightarrow 2x = 2y y' \)

\[
\begin{align*}
\frac{d}{dx} \left( 2x \right) &= \frac{d}{dx} \left( 2y y' \right) \\
2 &= 2 \left( y' y'' + y y''' \right) \\
\Rightarrow 1 &= \left( y'' \right)^2 + y y''' \text{ is a second-order ODE}
\end{align*}
\]

For which \( x^2 + 1 = y^2 \) is an (implicit) solution.

28. \( 2xy \, dx + (x^2 + y^2) \, dy = 0 \)

\[
\begin{align*}
\frac{\partial}{\partial y} \left( 2xy \right) &= 2x = 2x \\
\text{Since these are equal, } 2xy \, dx + (x^2 + y^2) \, dy = 0 \text{ is an exact ODE.}
\end{align*}
\]

29. \( 2y^2 \, dx + (1 + 2xy) \, dy = 0 \)

\[
\begin{align*}
\frac{\partial}{\partial x} \left( 2y^2 \right) &= 0 \\
\frac{\partial}{\partial x} \left( 1 + 2xy \right) &= 2y \\
\text{So this is not exact, but the form of this above suggests } h = h(y) \text{ will be an integrating factor.}
\end{align*}
\]

\[
\begin{align*}
h(y)2y^2 \, dx + h(y) \left( 1 + 2xy \right) \, dy &= 0 \\
\frac{\partial}{\partial x} \left( h(y)2y^2 \right) &= h(y)2y \\
h'(y)2y^2 + h(y)2y &= h(y)2y \\
\Rightarrow 2y^2 h' = -2y h &\Rightarrow \frac{h'}{h} = -\frac{1}{y} \\
\Rightarrow \int \frac{dh}{h} = -\int \frac{dy}{y} &\Rightarrow \ln h = -\ln y \\
&\Rightarrow h(y) = \frac{1}{y}.
\end{align*}
\]
So $2y \, dx + \left( \frac{1}{y} + 2x \right) \, dy = 0$ is **exact** (for $y \neq 0$).


\[
\int 2y \, dx = 2xy + c(y) = f(x,y),
\]

\[
f_y(x,y) = 2x + c'(y) = \frac{1}{y} + 2x \Rightarrow c'(y) = \frac{1}{y},
\]

\[
\Rightarrow c(y) = \int \frac{dy}{y} = \ln|y| + C.
\]

So $2xy + \ln|y| + C = 0$ is **an implicit solution**.

Now $y(2) = 1 \Rightarrow 2(2)(1) + \ln|1| + C = 0$

\[
\Rightarrow -4 = C, \quad \text{so}
\]

$2xy + \ln|y| - 4 = 0$ is **the particular solution**.

\[\begin{align*}
(2) & B'(t) = KB(t) \quad \text{where } B \text{ is the amount of bacteria, and } t \text{ is time (in days).} \\
(3) & B(0) = 500, \quad B(1) = 1500. \\
& B' = KB \Rightarrow \frac{dB}{B} = K \Rightarrow \int \frac{dB}{B} = \int KT \Rightarrow \ln|B| = KT + C \\
& \Rightarrow B(t) = Ce^{KT} \\
& B(0) = 500 \Rightarrow C = 500, \\
& B(1) = 1500 \Rightarrow 500e^K = 1500 \Rightarrow e^K = 3, \quad \text{so} \\
& B(t) = 500 \cdot 3^t. \\
(4) & B(T) = 5000 \Rightarrow 500 \cdot 3^T = 5000 \Rightarrow 3^T = 10, \\
& \text{so } T = \frac{\ln 10}{\ln 3}.
\end{align*}\]

\[\begin{align*}
(5) & y' = y + e^x \Rightarrow y' - y = e^x, \quad \text{This is linear, so} \\
& h(x) = e^{-x} \Rightarrow dx = e^{-x} \\
& \text{Thus } y(x) = e^x \int e^{-x} e^x \, dx = e^x \int dx = e^x [x + C], \\
& \text{and therefore } y(x) = xe^x + Cx.
\end{align*}\]

\[\begin{align*}
(6) & y^3 + xy + x + 1 = 0 \Rightarrow 3y^2y' + y + xy' + 1 = 0, \\
& \text{so this is a first-order ODE for which } y^3 + xy + x + 1 = 0 \text{ is an implicit solution.}
\end{align*}\]
53. \( m v' = F_g + F_a = mg - 10v \)

\( m = 10, \ g = 9.8 \text{ m/s}^2, \)

so \( 10v' = 10 \cdot 9.8 - 10v \)

\[ \Rightarrow v' = 9.8 - v, \ \text{AND} \ v(0) = 0. \]

\[ \int \frac{dv}{9.8 - v} = \int dt \Rightarrow - \int \frac{dv}{v - 9.8} = \int dt, \]

so \(-\ln |v - 9.8| = t + C \Rightarrow v - 9.8 = Ce^{-t}, \)

\[ \Rightarrow v(t) = Ce^{-t} + 9.8. \]

Since it was dropped (AND NOT THROWN), \( v(0) = 0. \)

\[ \Rightarrow C + 9.8 = 0 \Rightarrow C = -9.8. \]

\[ \Rightarrow v(t) = 9.8(1 - e^{-t}). \]

54. \( y = \left( x + \frac{x^2}{y^2 + 1} \right) y' \Rightarrow (y^2 + 1)y \frac{dy}{dx} + \left( -x(y^2 + 1) - x^2 \right) \frac{dy}{dx} = 0 \)

\[ \frac{2}{3} \sqrt{3y^2 + 1} \]

\[ 3y^2 + 1 \neq -y^2 - 1 - 2x \]

So it is not exact.

\[ h(x) = \left( y^2 + 1 \right) y \frac{dy}{dx} + h(x) \left( -x(y^2 + 1) - x^2 \right) \frac{dy}{dx} = 0 \]

\[ \frac{3}{\sqrt{3}} \]

\[ h(x) = h'_{y'}(x) = h'(x) \left( -x(y^2 + 1) - x^2 \right) + h(x) \left( -4y^2 - 1 - 2x \right) \]

\[ \Rightarrow h'(x) \left( x(y^2 + 1) + x^2 \right) = h(x) \left( -4y^2 - 1 - 2x \right) \]

\[ \Rightarrow \frac{h'(x)}{h(x)} = \frac{-4y^2 - 1 - 2x}{x(y^2 + 1) + x^2} \]

// Can't obviously get rid of \( y \), so let's try \( h \) (ly) instead.
\[ h(y) (y^2+1) \, y \, dx + h(y) (-x (y^2+1) - x^2) \, dy = 0 \]

\[ \Rightarrow \frac{\partial}{\partial y} (\frac{y^2}{y^2+1}) = \frac{\partial}{\partial x} (\frac{x}{y^2+1}) \]

**This also is going to be tough to solve, we should revert to the original and see if we are missing anything:**

\[ y = (x + \frac{x^2}{y^2+1}) y' \]

\[ \Rightarrow y \, dx + \left(-x - \frac{x^2}{y^2+1}\right) \, dy = 0 \]

Let's not try and "simplify" it by multiplying by \( y^2+1 \), as we did before.

\[ 1 \neq -1 - \frac{x}{y^2+1} \]

Not exact, but let's try \( h = h(x) \):

\[ h(x) y \, dx + h(x) \left(-x - \frac{x^2}{y^2+1}\right) \, dy = 0 \]

\[ \Rightarrow h(x) \left(x + \frac{x^2}{y^2+1}\right) = h(x) \left(-2 - \frac{2x}{y^2+1}\right) \]

\[ \Rightarrow \frac{h'(x)}{h(x)} = \frac{-2 \left(1 + \frac{x}{y^2+1}\right)}{x \left(1 + \frac{x}{y^2+1}\right)} = \frac{-2}{x} \]

So this works, and solving yields:

\[ \int \frac{dh}{h} = \int \frac{-2}{x} \, dx \Rightarrow \ln h = -2 \ln x \Rightarrow h(x) = \frac{1}{x^2} \]

So \[ \frac{y}{x^2} \, dx + \left(-\frac{1}{x} - \frac{1}{y^2+1}\right) \, dy = 0 \] is exact.

\[ f(x,y) = \int \frac{y}{x^2} \, dx = \frac{-y}{x} + C(y), \]

\[ f_y(x,y) = \frac{-1}{x} + C'(y) = \frac{-1}{x} - \frac{1}{y^2+1} \Rightarrow C'(y) = -\frac{1}{y^2+1} \]

\[ \Rightarrow C(y) = \int \frac{-1}{y^2+1} \, dy = -\arctan y + C, \text{ so} \]

\[ \frac{-y}{x} - \arctan y + C = 0 \text{ is an implicit solution.} \]
This is a good example to illustrate that, in general, finding an integrating factor can be (much) more difficult, depending on the form of the ODE.

\[ 2x \, dx + 2y \, dy = 0 \]
\[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) \]
\[ 0 = 0 \]

This is an exact ODE, because \( 0 \) is continuous everywhere.

\[ y \, y' + y^2 + 1 = x \sqrt{y^2 + 1} \]

// NOTE THE \( y \) MISTAKE, TYPO IN ORIGINAL REVIEW. \n\[ \text{Let } u = \sqrt{y^2 + 1}, \text{ as suggested, then } u' = \frac{y}{\sqrt{y^2 + 1}} y', \text{ so } \]
\[ u \, u' = y \, y', \text{ and thus } \]
\[ u \, u' + u^2 = x \, u. \]

Since \( u = \sqrt{y^2 + 1} > 0 \), we divide by \( u \) (no side-condition of \( u \) if needed):
\[ \Rightarrow u' + u = x, \text{ this is linear, and } \]
\[ u(x) = e^{-\int dx} \int e^{\int dx} x \, dx = e^{-x} \int x \, e^x \, dx \]
\[ = e^{-x} [xe^x - \int e^x \, dx + C] = e^{-x} [xe^x - e^x + C] \]
\[ \Rightarrow \sqrt{y^2 + 1} = x - 1 + Ce^{-x} \text{ is an implicit solution.} \]

\[ (y^2 + 1) (2yy' + 1) = 1 \]

This hint suggests a substitution, an initial guess of \( u = y^2 + x \) \[ \Rightarrow u' = 2yy' + 1, \text{ so we have } \]
\[ u \, u' = 1 \Rightarrow \frac{\partial}{\partial u} \left( \frac{\partial}{\partial x} \right) \]
\[ \Rightarrow (y^2 + x)^2 = 2x + C \text{ is an implicit solution.} \]
(iii) \( y = xe^x \Rightarrow y' = e^x + xe^x, \ y'' = e^x + e^x + xe^x = 2e^x + xe^x. \)

Now,

\[
y'' + y = (2e^x + xe^x) + (e^x) = 3e^x + xe^x \neq x,
\]

so this is not a solution of \( y'' + y = x. \)

(iv) \( y' = 3x + xy \Rightarrow \frac{dy}{3+y} = \frac{dx}{x} \Rightarrow \ln |3+y| = \frac{x^2}{2} + C \)

\( \Rightarrow 3+y = Ce^{\frac{x^2}{2}} \Rightarrow y(x) = Ce^{\frac{x^2}{2}} - 3. \)

(Note if \( y + 3 = 0 \Rightarrow y(x) = -3, \) which is a solution included above with \( C = 0. \))

Now if \( y(0) = 1 \Rightarrow C - 3 = 1 \Rightarrow C = 4, \) so \( y(x) = 4e^{\frac{x^2}{2}} - 3 \) is the particular solution.

(v) \( \sin^2 x + \cos^2 y = 0 \Rightarrow \frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(0) \)

\( \Rightarrow 2\sin x \cos x - 2\sin y \cos y y' = 0, \)

This is enough, but it can be simplified to \( y' = \frac{\sin 2x}{\sin 2y}, \) for \( \sin 2y \neq 0. \)

(vi) \( \cos(x+y)dx + \cos(x+y)dy = 0 \)

\( \frac{dy}{dx} = \frac{-\sin(x+y)}{\cos(x+y)} \)

(THIS IS EXACT)

\( \int \cos(x+y)dx = \sin(x+y) + C_1(y), \)

\( \int \cos(x+y)dy = \sin(x+y) + C_2(x) \)

so, taking \( C_1(y) = C_2(x) = C, \sin(x+y) + C = 0 \) is an implicit solution. We can solve this explicitly:

\( x + y = \arcsin(C) \Rightarrow y(x) = -x + \arcsin(C). \)
\[ y' = x + 3y, \quad x + 3y = c \Rightarrow y = -\frac{x}{3} - \frac{c}{3} \]

<table>
<thead>
<tr>
<th>Slope</th>
<th>Isocline</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( y = -\frac{1}{3}x - \frac{2}{3} )</td>
</tr>
<tr>
<td>-1</td>
<td>( y = -\frac{1}{3}x - \frac{1}{3} )</td>
</tr>
<tr>
<td>0</td>
<td>( y = -\frac{1}{3}x )</td>
</tr>
<tr>
<td>1</td>
<td>( y = -\frac{1}{3}x + \frac{1}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( y = -\frac{1}{3}x + \frac{2}{3} )</td>
</tr>
</tbody>
</table>

\( y = \frac{1}{3}x - \frac{c}{3} \) corresponds to slope \(-\frac{1}{3}\), and the isocline has slope \(-\frac{1}{3}\), so this actually is a solution, as we verify:

\[ y(x) = \frac{1}{3}x - \frac{1}{3} \Rightarrow y'(x) = -\frac{1}{3} \]

\(-\frac{1}{3} = x + 3\left(\frac{1}{3}x - \frac{1}{3}\right) \sqrt{\quad}\)

\((y + 2xy\cos^2(xy))dx + (x + x^2\cos^2(xy))dy = 0 \quad \text{(original)}\)

Using the integrating factor \(\sec^2(xy)\), we get

\((y\sec^2(xy) + 2xy)dx + (x\sec^2(xy) + x^2)dy = 0\)

This should be exact, and we can solve it:

\[ f(x,y) = \int x\sec^2(xy) + x^2\ dy = \tan(xy) + x^2y + C(x), \]

\[ f_x(x,y) = y\sec^2(xy) + 2xy + C'(x) = y\sec^2(xy) + 2xy \]

\[ \Rightarrow C'(x) = 0 \Rightarrow C(x) = C, \quad \text{so} \]

\[ y\sec^2(xy) + 2xy + C = 0 \quad \text{is an implicit solution.} \]

R0 = # of rabbits, \( t \) = time (in days).
R0 = R0
R(50) = 2R0
R1 = kR \Rightarrow \int_0^1 R = \int_0^1 kdt \Rightarrow \ln |R| = kt + C \Rightarrow R(t) = Ce^{kt},
R0 = |R0 \Rightarrow C = R0, \quad k50 = 2R0 \Rightarrow e^{k50} = 2R0.
We want \( R(180) = 400, \Rightarrow R0 \cdot 2^{\frac{180}{50}} = 400 \]
\[ \Rightarrow R0 = 400 \cdot 2^{-\frac{180}{50}}. \]
Of course, we can't start with a non-integral # of rabbits, which is why the question asks for an approximation, which \( 400 \cdot 2^{-1/5} \) is. The actual minimum # of rabbits can be expressed as \([400 \cdot 2^{-1/5}]\). (Use the "ceiling" function)

\[ y' + \frac{1}{x} y = 4x^2 \]

is linear:

\[ y(x) = e^{-\int \frac{1}{x} dx} \int e^\frac{1}{x} 4x^2 \, dx = \frac{1}{x} \int 4x^2 \, dx = \frac{1}{x} \left[ x^3 + C \right] = x^3 + \frac{C}{x} \]

There is no particular solution with \( y(0) = 1 \). Let's see that this does not violate the existence theorem:

\[ y' = 4x^2 - \frac{y}{x} = \frac{4x^2 - y}{x} \]

Discontinuous at \((0,1)\). ✓

So, we aren't guaranteed a solution, and we showed that there isn't one.

\[ y' = -\frac{x}{y} \Rightarrow y \, y' = -x \Rightarrow \int y \, dy = -\int x \, dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C \]

\[ \Rightarrow y^2 + x^2 = C \] is an implicit solution.

\[ \text{17} \]

\[ m \ddot{v} = mg - 5v \]

\[ \Rightarrow 5 \dot{v} = 5g - 5v \Rightarrow \dot{v} = g - \frac{v}{5} \]

And \( v(0) = -30 \)

So \[ \int \dot{v} = g - \frac{v}{5} \]

\[ v = g - \frac{v}{5} \Rightarrow \frac{dv}{v-9.8} = -\frac{dt}{5} \Rightarrow \ln |v-9.8| = -t + C \]

\[ \Rightarrow v-9.8 = Ce^{-t} \Rightarrow v(t) = Ce^{-t} + 9.8 \]

And \( v(0) = -30 \Rightarrow C + 9.8 = -30 \Rightarrow C = -39.8 \).

\[ v(t) = -39.8 e^{-t} + 9.8 \]

The ball reaches max height when \( v(t) = 0 \),

\[ \Rightarrow -39.8 e^{-t} + 9.8 = 0 \Rightarrow e^{-t} = \frac{9.8}{39.8} \Rightarrow t = \ln \left( \frac{398}{98} \right) \text{ seconds.} \]
\[ \frac{dT}{dt} = -K(T(t) - M(t)) \]

\[ T(t) = -K(T - 3\sin(\pi t) + 10), \]

\[ T(0) = 60 \]

\[ T(1) = 40 \]

(b) \[ T' = -KT + K3 \sin(\pi t) + K \cdot 10 \]

\[ \Rightarrow T' + KT = 3K \sin(\pi t) + 10K \] is linear,

\[ h(t) = e^{\int -K dt} = e^{-Kt} \]

\[ T(t) = e^{-Kt} \int e^{Kt}(3K \sin(\pi t) + 10K) \, dt \]

\[ \int e^{Kt} \sin(\pi t) \, dt \quad \text{let } u = \sin(\pi t) \quad dv = e^{Kt} \, dt \]

\[ du = \pi \cos(\pi t) \, dt \quad v = \frac{1}{K} e^{Kt} \]

\[ = \frac{e^{Kt}}{K} \sin(\pi t) - \frac{\pi}{K} \int e^{Kt} \cos(\pi t) \, dt + C \quad \text{let } w = \cos(\pi t) \]

\[ dw = -\pi \sin(\pi t) \, dt \]

\[ = \frac{e^{Kt}}{K} \sin(\pi t) - \frac{\pi}{K} \left( \frac{e^{Kt}}{K} \cos(\pi t) + \frac{\pi}{K} \int e^{Kt} \sin(\pi t) \, dt \right) + C \]

\[ \Rightarrow (1 + \frac{\pi^2}{K^2}) \int e^{Kt} \sin(\pi t) \, dt = \frac{e^{Kt}}{K} \left( \sin(\pi t) - \frac{\pi}{K} \cos(\pi t) \right) + C \]

\[ \Rightarrow \int e^{Kt} \sin(\pi t) \, dt = \frac{e^{Kt} \left( K \sin(\pi t) - \pi \cos(\pi t) \right)}{K^2 + \pi^2} + C \]

\[ \text{Therefore} \]

\[ T(t) = e^{-Kt} \left[ \frac{3Ke^{Kt} \left( K \sin(\pi t) - \pi \cos(\pi t) \right)}{K^2 + \pi^2} + 10e^{Kt} + C \right] \]

\[ = \frac{3K(K \sin(\pi t) - \pi \cos(\pi t))}{K^2 + \pi^2} + 10 + Ce^{-Kt} \]

Now \[ T(0) = \frac{3K\sin(0)}{K^2 + \pi^2} + 10 + C = 60 \Rightarrow C = 50 + \frac{3K\pi}{K^2 + \pi^2} \]

\[ T(1) = \frac{3K\sin(\pi)}{K^2 + \pi^2} + 10 + Ce^{-K} = 40 \Rightarrow Ce^{-K} = 30 - \frac{3K\pi}{K^2 + \pi^2} = 80 - C \]

\[ \Rightarrow C = \frac{80}{1 + e^{-K}} \]

Solving the system \[ \left\{ \begin{array}{l} C = 50 + \frac{3K\pi}{K^2 + \pi^2} \\ C = \frac{80}{1 + e^{-K}} \end{array} \right. \]

Can be done numerically (by computer), but on an exam (where you cannot use a calculator) I would give you K and C, or make them easier to calculate.

You may omit solving for K and C here.
\[ T(\tau) = \frac{-3\tau t}{K^2 + \pi^2} + 10 + Ce^{-2\tau t} \]

"Again, on an exam I would give you K and C, or make them easier to calculate.

\[ \text{The answer to this is yes, as } Ce^{-2\tau t} \rightarrow 0 \text{ as } t \rightarrow \infty, \]

\[ \frac{2K(K\sin(\pi t) - \pi\cos(\pi t))}{K^2 + \pi^2} \] oscillates positive/negative, so we could find a \( t_0 \) where \( T(t_0) < 0 \).

"This question is a little too nuanced for an exam.

\[ y' + \frac{1}{x} y = y^2 \]

INTEGRATING FACTOR \( y^{-2} \) \( \text{if } y \neq 0 \)

\[-y^{-2} y' + \frac{1}{x} y^{-1} = 1\]

Let \( u = y^{-1} \Rightarrow u' = -y^{-2} y' \), so

\[ u' + \frac{1}{x} u = 1\]

\[ \Rightarrow u(x) = e^{-\frac{\int}{x} \int} e^{\frac{1}{x}} dx = \frac{1}{x} \int x dx = \frac{1}{x} [\frac{x^2}{2} + C] \]

\[ \Rightarrow y(1) = 1 \Rightarrow \frac{2}{1 + C} = 1 \Rightarrow C = 1 \]

\[ \text{so } y(x) = \frac{2x}{x^2 + 1} \text{ is the particular solution.}\]

\[ S = \text{AMOUNT OF SALT (IN KG)}, t = \text{TIME (IN MINUTES)} \]

\[ S' = \text{RATE IN - RATE OUT} \]

\[ = k - \frac{S}{200} \text{, WHERE } K \text{ IS THE UNKNOWN INCOMING SALT CONCENTRATION.} \]

\[ \begin{cases} S' = k - \frac{S}{200} \\ S(0) = 0 \\ S(1) = 10 \end{cases} \]

\[ \int \frac{dS}{k - \frac{S}{200}} = \int dt \]

Let \( u = k - \frac{S}{200} \Rightarrow du = -\frac{1}{200} dS \Rightarrow -200 du = dS \]

\[ -200 \ln |u| = t + C \Rightarrow k - \frac{S}{200} = Ce^{-t/200} \]

\[ \Rightarrow S(t) = Ce^{-t/200} + 200K \]

\[ S(0) = C + 200K = 0 \]

\[ S(1) = Ce^{-1/200} + 200K = 10 \]
\[ C = -200k \Rightarrow 200k \left( 1 - e^{-\frac{t}{200}} \right) = 10 \]

\[ \Rightarrow k = \frac{10}{200 \left( 1 - e^{-\frac{t}{200}} \right)}, \quad C = \frac{-10}{1 - e^{-\frac{t}{200}}} \]

Thus the concentration of the incoming salt water is

\[ \frac{10}{200 \left( 1 - e^{-\frac{t}{200}} \right)} \text{ kg/L} \]

\[ S(t) = \frac{10}{1 - e^{-\frac{t}{200}}} \left( - e^{-\frac{t}{200}} + 1 \right) = 10 \frac{1 - e^{-\frac{t}{200}}}{1 - e^{-\frac{t}{200}}} \]

\[ S'(t) = \frac{10}{200 \left( 1 - e^{-\frac{t}{200}} \right)} e^{-\frac{t}{200}} = ke^{-\frac{t}{200}} > 0, \quad \text{so} \]

The amount of salt is monotonically increasing. Therefore, the least upper bound is

\[ \lim_{t \to 00} S(t) = \frac{10}{1 - e^{-\frac{t}{200}}} \left( = 200k \right) \text{ kg/L} \]