Suggested Exercises (K - 1.1 - 1.4, T&P - 1-3, 5, 6C, 9-10)

Solve the following differential equations using separation of variables:
1. \( x \, dy - y \, dx = 0 \)
2. \( \frac{dr}{d\theta} = r \tan \theta \)
3. \( \sin x \cos 2y \, dx + \cos x \sin 2y \, dy = 0, \quad y(0) = \pi/2 \)
4. \( y' \cosh^2 x = \sin^2 y, \quad y(0) = \frac{\pi}{2} \)

Solve the following exact differential equations:
5. \( (e^x \sin y + e^{-y}) \, dx - (xe^{-y} - e^x \cos y) \, dy = 0 \)
6. \( (2x + y \cos x) \, dx + (2y \sin x - \sin y) \, dy = 0 \)
7. \( (4x^3 - \sin x + y^3) \, dx - (y^2 + 1 - 3xy^2) \, dy = 0 \)

Solve the following differential equations by finding an integrating factor:
8. \( (x^2 + y^2 + x) \, dx + (xy) \, dy = 0 \)
9. \( (x^4y^2 - y) \, dx + (x^2y^4 - x) \, dy = 0 \)
10. \( (y^2 - 3xy - 2x^2) \, dx + (xy - x^2) \, dy = 0 \)

Solve the following differential equations using any method:
11. \( (e^x \sin y + e^{-y}) \, dx - (xe^{-y} - e^x \cos y) \, dy = 0 \)
12. \( 3(y + x)^2 \, dx + x(3y + 2x) \, dy = 0 \)
13. \( y \, dy + x \, dx = 3xy^2 \, dx, \quad y(2) = 1 \)
14. \( y' = e^{2x-1}y^2 \)
15. \( xy' = 1 + x^2 + y^2 + x^2y^2 \)

Solve the following problems by setting up and solving a differential equation.
16. If a wet sheet in a dryer loses its moisture at a rate proportional to its moisture content, and if it loses half of its moisture during the first 10 min of drying, when will it be practically dry, say, when will it have lost 99\% of it’s moisture?

17. If a strain of bacteria grows at a rate proportional to the amount present and if the population doubles in one hour, by how much will it increase at the end of two hours?
18. A tank initially contains 100 gal of fresh water. Brine whose salt concentration is 2 lb/gal flows into the tank at the rate of 2 gal/min. The mixture flows out at the same rate.

(a) Find the salt content of the brine at the end of 100 min.
(b) At what time will the salt concentration reach 1 lb/gal?
(c) Could the salt content of the brine ever reach 200 lb?

Challenging Problems
I will not select from these for the quiz, but you may want to try them out of curiosity or to test yourself.

19. A woman wearing a parachute jumps from a great height. The combined weight of woman and parachute is 136 pounds. Let \( v(t) \) denote her speed (in feet per second) at time \( t \) seconds after falling. During the first 10 seconds, before the parachute opens, assume the air resistance is \( \frac{1}{2}v(t) \) pounds. Thereafter, while the parachute is open, assume the resistance is 10\( v(t) \) pounds. Assume the acceleration of gravity is 32 ft/sec\(^2\) and find explicit formulas for the speed \( v(t) \) at time \( t \).

(a) How far does the woman fall before opening the parachute?
(b) How high should the woman have jumped from in order to land at a velocity of 1 foot / sec (which is still quite fast)?
(c) What is the terminal velocity of the woman if she did not open the parachute? (Terminal velocity is the highest velocity attainable by an object as it falls through the air.)
(d) What is the terminal velocity of the woman after having opened the parachute?

20. Use the method discussed in class to find a formula for an integrating factor \( \mu(u) \), where \( u = y/x \). (If you need a hint, see page 89 of T&P.)

21. Denote by \( y = f(t) \) the amount of a substance present at time \( t \). Assume it disintegrates at a rate proportional to the amount present. If \( n \) is a positive integer, the time \( T \) for which \( f(T) = f(0)/n \) is called the 1/\( n \)th life of the substance.

(a) Prove that the 1/\( n \)th life is the same for every sample of a given material, and compute \( T \) in terms of \( n \) and the decay constant \( k \).
(b) If \( a \) and \( b \) are given, prove that \( f \) can be expressed in the form

\[
f(t) = f(a)^{w(t)} f(b)^{1-w(t)}
\]

and determine \( w(t) \). This shows that the amount present at time \( t \) is a weighted geometric mean of the amounts present at two instants \( t = a \) and \( t = b \).
In class we hypothesized that, given an ODE in the form $M \, dx + N \, dy = 0$, if $M$ and $N$ were “jointly symmetric” with respect to $x$ and $y$, that is, if $M(x, y) = N(y, x)$, then we seemed to be able to find an integrating factor of the form $\mu(xy)$. If you think this is true in general, prove it. If you think this is false, try and find a counter-example, and if you are able to find a counter-example, see if you could prove something weaker (or a modification of our statement).