Research Statement:
Combinatorial Methods in Algebra and Geometry

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My research is at the intersection of algebra, geometry, and combinatorics. My dissertation work under Pramod Achar involved a deep study of the singularities of certain topological spaces through the lens of perverse sheaves. Concurrently, I began working with Greg Muller on cluster algebras, a subject that is related to a large number of areas in mathematics, but which has a somewhat easier point of entry due to its computable examples.

This early blend of, almost diametrically opposed, styles of math shaped my research direction. My work gravitated toward interpreting complicated topological or algebraic questions as combinatorial ones, which lend themselves to explicit computations. Each of my projects has this theme in one way or another. My research statement is divided into three broad areas: matroids, cluster algebras, and representation theory. The projects described in each subsection are outlined below.

Section 1.1 This is an ongoing project, joint with Tom Braden (UMass) and Nicholas Proudfoot (UOregon), which aims to introduce topological and Hodge-theoretic techniques into the field of matroid theory. We hope to prove the non-negativity of Kazhdan–Lusztig polynomials of matroids.

Section 1.2 This is a project, joint with Eric Bucher (Michigan State), which gives a cancellation-free antipode formula for the localization-contraction combinatorial Hopf algebra for matroids.

Section 2.1 This is a project, joint with Greg Muller (UOklahoma), that develops an algebro-geometric algorithm which gives a presentation for upper cluster algebras in terms of generators and relations. This algorithm is computable in finite-time, and has been implemented into Sage.

Section 3.1 This is a project, joint with Alexander Garver (LaCIM), Kiyoshi Igusa (Brandeis), and Jonah Ostroff (UWashington), which introduces a combinatorial model, called strand diagrams, for exceptional sequences of type $A$ quiver representations. Recently, Alejandro Morales (UMass) and I have started looking for a formula for the counting linear extensions of certain posets arising from strand diagrams.

Section 3.2 This is an ongoing project, joint with Pramod Achar (Louisiana State) and Maitreyee Kulkarni (Louisiana State), which hopes to introduce a combinatorial model, called triangular arrays, for computing the Fourier–Sato transform on type $A$ quiver representation varieties.

Section 3.3 This project was my dissertation work which gives a geometric, functorial relationship between representations of an algebraic group and representations of the corresponding Weyl group at the level of mixed, derived categories of sheaves on the affine Grassmannian and nilpotent cone of the Langlands dual group.

Remark Each of the broad sections below can be read independently, so the reader is invited to pick his or her favorite subject and go directly to that page.

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1 Matroids

A matroid is a gadget which generalizes the notion of linear (in)dependence of vectors in a vector space; more specifically, a matroid is an object \( M \) defined by a finite set \( I \) together with a collection of subsets satisfying some “dependence axioms” (see [Oxl11]). Examples include vector matroids, matroids arising from hyperplane arrangements, and matroids arising from graphs.

Every matroid \( M \) is equipped with a graded poset called its lattice of flats \( L(M) \). For example, the linear subspace \( V := (x_1 + x_2 + x_3 = 0) \) in \( \mathbb{C}^{I=\{1,2,3\}} \) defines a hyperplane arrangement (hence a matroid) after intersecting it with the coordinate hyperplanes. All possible intersections of the hyperplanes are the flats, and the partial order is given by containment, as illustrated below.

For each flat \( F \in L(M) \), we can define two new matroids:

- \( M_F \) is a certain matroid on \( F \) called the localization of \( M \) at \( F \), and
- \( M^F \) is a certain matroid on \( I \setminus F \) called the contraction of \( M \) at \( F \).

The figure below shows an example (here \( M \) is the uniform matroid of rank 3 on \( I = \{1, 2, 3, 4\} \)) of these two constructions in terms of the lattice of flats \( L(M) \).

The localization and contraction matroids will play a key role in the two problems described below.

1.1 Kazhdan–Lusztig polynomials of matroids

In this project, I propose introducing topological and Hodge-theoretic techniques into the field of matroid theory. This will be done as part of an ongoing research plan to prove that Kazhdan–Lusztig (KL) polynomials of matroids have non-negative coefficients. Just as KL theory for Coxeter groups guided the development of geometric representation theory over thirty-five years, I expect that this work will strengthen matroid theory’s connection with other areas and be indicative of the direction in which matroid theory will develop in the future.

In [EPW16], Elias, Proudfoot, and Wakefield associated to each matroid \( M \) a polynomial \( P_M(t) \) called the Kazhdan–Lusztig (KL) polynomial of \( M \). These polynomials share many analogies with the classical KL polynomials [KL79] for Coxeter groups, but exhibit some interesting differences. Both types of polynomials have a purely combinatorial definition—while KL polynomials for Coxeter groups are defined in terms of more elementary polynomials called \( R \)-polynomials, every matroid has a characteristic polynomial \( \chi_M(t) \) which plays this role. In the classical setting, Polo showed that every polynomial with non-negative coefficients and constant term 1 occurs as a KL polynomial [Pol99] for some symmetric group \( \mathfrak{S}_n \). In stark contrast, it is...
conjectured that the $P_M(t)$ are real-rooted \cite{GPY16}, which implies that their coefficients form a log-concave sequence.

When the Coxeter group is in fact a finite Weyl group, there is a geometric interpretation of the classical KL polynomials. They are the intersection cohomology Poincaré polynomials of stalks of Schubert varieties, which implies non-negativity of their coefficients. In a similar way, when $M$ is representable (i.e. arises from a hyperplane arrangement), Elias, Proudfoot, and Wakefield identified $P_M(t)$ with the intersection cohomology Poincaré polynomial of a variety, built from the hyperplane arrangement, called the reciprocal plane $X(V)$.$^1$

**Theorem 1.1.1** (\cite{EPW16}). If $M$ is a representable matroid, then $P_M(t) = \sum_{i \geq 0} t^i \dim H^{2i}(X(V))$.

In this way, the $P_M(t)$ have non-negative coefficients when $M$ is representable. The question of non-negativity of the KL polynomials for arbitrary Coxeter groups was conjectured in \cite{KL79}, but remained unsolved for thirty-five years. In 2014, Elias and Williamson settled this question in the affirmative by using sophisticated diagrammatic combinatorics \cite{EW17, EW16} to give an algebraic proof of the decomposition theorem \cite{BBD82} via the Hodge-theoretic properties of Soergel bimodules \cite{EW14}. Braden, Proudfoot, and I are interested in proving the analogous conjecture for matroids.

**Conjecture 1.1.2** (\cite{EPW16}). For an arbitrary matroid $M$, the coefficients of $P_M(t)$ are nonnegative.

Despite the simplicity of this statement, just as in the classical setting, investigating this conjecture leads to deep connections relating topology, combinatorics, and representation theory. A recent example of work in the same vein is that of Adiprasito, Huh, and Katz \cite{AHK15}. They proved, using deep topological arguments, that the coefficients of the characteristic polynomial $\chi_M(t)$ of an arbitrary matroid form a log-concave sequence; thereby settling a long-standing conjecture of Rota, Heron, and Welsh \cite{Rot71, Her72, Wel76}.

We include here, for convenience, a table summarizing the above discussion, and more.

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**Combinatorial intersection cohomology of matroids** The usual plan of attack for proving statements like Conjecture 1.1.2 is to replace intersection cohomology sheaves with combinatorial objects and to prove that various Hodge-theoretic properties (including the Decomposition Theorem) carry over to the combinatorial setting.

Together with Tom Braden and Nicholas Proudfoot, I defined a combinatorial category $\mathcal{C}(M)$ which models constructible sheaves on the reciprocal plane. Inside $K^b(\mathcal{C}(M))$, we defined an object $\text{IC}^\bullet_M$ which should be understood as the intersection cohomology of an arbitrary matroid (analogous to an indecomposable Soergel bimodule of Elias–Williamson). After defining combinatorial stalk functors $s_F$ inside $K^b(\mathcal{C}(M))$ for each $F \in L(M)$, we arrive at the following conjecture, which should be viewed as an analog of Theorem 1.1.1 for arbitrary matroids (when no geometry is available).

**Conjecture 1.1.3** (Braden–M.–Proudfoot 2017). For an arbitrary matroid $M$, the KL polynomial $P_M(t) = \sum_{i \geq 0} t^i \dim H^i(s_0(\text{IC}^\bullet_M))_i$.

Verifying the above conjecture about the complex of graded vector spaces $s_0(\text{IC}^\bullet_M)$ would immediately imply Conjecture 1.1.2. As stated earlier, proving this conjecture generally involves establishing purity (in the sense of mixed Hodge theory). That is, we must prove cohomology is concentrated “on the diagonal”.$^1$

$^1$ The definition of the reciprocal plane $X(V)$ depends on a linear space $V$, like in the example in Section [4].
Conjecture 1.1.4 (Braden–M.–Proudfoot 2017). The combinatorial intersection cohomology sheaf $\IC^\bullet_M$ of a matroid $M$ satisfies $H^i(s^0(\IC^\bullet_M)) = 0$ unless $i = j$.

Here $s^0$ is a certain combinatorial costalk functor defined on $K^b(C(M))$.

It was already known, for an arbitrary matroid $M$, that the linear coefficient of $P_M(t)$ is non-negative; however, we are able to give a different proof using $\IC^\bullet_M$. Furthermore, using this machinery we proved a new result, that the quadratic coefficient of $P_M(t)$ is non-negative.

Theorem 1.1.5 (Braden–M.–Proudfoot 2017). Let $M$ be an arbitrary matroid. The quadratic coefficient of $P_M(t)$ is non-negative.

Matroidal Bott–Samelson modules Our proof that the quadratic coefficient of $P_M(t)$ is non-negative for an arbitrary matroid $M$ uses ad-hoc methods and requires a long and complicated cohomology patching argument; thus, it will probably not generalize to all coefficients.

A more holistic approach is to mirror Elias–Williamson’s construction of Bott–Samelson (BS) bimodules. For this purpose, in the representable case, we prefer to work with a projective variety $Y(V)$ defined to be the closure of $V$ inside $(\mathbb{P}^1)^J$, since it has a stratification by affine cells. We have defined a certain resolution $\pi_V : \widetilde{Y(V)} \to Y(V)$ where the variety $\widetilde{Y(V)}$ is given by blowing up $Y(V)$ at various strata$^2$.

Braden, Proudfoot, and I constructed an object $\Omega^\bullet_M$ in $K^b(C(M))$ that conjecturally plays the role of the pushforward sheaf $(\pi_V)_*\IC^\bullet_Y$ (analogous to the BS bimodules). These $\Omega^\bullet_M$ have a simpler structure than the $\IC^\bullet_M$, so their purity (Conjecture 1.1.4 with $\IC^\bullet_M$ replaced by $\Omega^\bullet_M$) will be easier to verify. To complete the proof of Conjecture 1.1.2 using this approach, we must show that $\Omega^\bullet_M$ contains all $\IC^\bullet_M$’s as indecomposable summands. That is, we must prove a “Decomposition Theorem” when $k$ is a field of characteristic zero.

Conjecture 1.1.6 (Braden–M.–Proudfoot 2017). The object $\Omega^\bullet_M$ has a decomposition

$$\Omega^\bullet_M \cong \bigoplus_{F \in L(M)} \IC^\bullet_{M_F} \otimes W_F.$$ 

where $M_F$ is the localization of $M$ at the flat $F$ (as in Section 7), and $W_F$ is a graded vector space keeping track of the multiplicity.

However, the Decomposition Theorem of Conjecture 1.1.6 is notoriously difficult in other settings (see for toric varieties and for Coxeter groups).

The strategy for proving Conjecture 1.1.6 usually involves “combinatorializing” an argument of DeCataldo and Migliorini (2002), which shows that the “Lefschetz package” implies the Decomposition Theorem of (2002) for smooth projective varieties. The simple construction of the BS bimodules allowed Elias and Williamson to develop a diagrammatic calculus to compute morphisms between all BS bimodules (2017). Using this intuition, they proved an analog of Conjecture 1.1.6, yielding the analog of Conjecture 1.1.2 for KL polynomials of Coxeter groups. We will investigate the analogous question in the matroidal setting.

Problem 1.1.7 (Braden–M.–Proudfoot). Find generators and relations for the endomorphism ring

$$\text{End}^\bullet_{K^b(C(M))} \left( \bigoplus_{F \in L(M)} \Omega^\bullet_{M_F} \right).$$

As a final remark, we note that recently Huh and Wang have “combinatorialized” and proven the Lefschetz package for the “Chow ring” $A^\bullet(M)$ of a matroid $M$ (2016). If $M$ is representable, then $A^\bullet(M)$ is the cohomology of a variety called the wonderful compactification of $Y(V)$. Since it is defined by performing a further series of blowups of $\widetilde{Y(V)}$, it may be possible to use these results in verifying Conjecture 1.1.6 in our setting.

$^2$ The reader familiar with the Coxeter group setting should interpret this as an analog of a BS resolution of a Schubert variety.
1.2 Antipodes of matroid Hopf algebras

Many combinatorial structures can be used as building blocks to construct graded connected Hopf algebras. Examples come from graphs, simplices, polynomials, Young tableaux, and many more. In this project, we focus on finding a cancellation-free antipode formula for the localization-contraction Hopf algebra of matroids (first introduced by Schmitt [Sch94]).

When a Hopf algebra arises from combinatorial objects, it is often referred to as a combinatorial Hopf algebra (see [GR16] for an excellent treatment of Hopf algebras, especially as they appear in combinatorics). Motivations for studying these algebras appear throughout many diverse areas of mathematics, including: combinatorics, representation theory, mathematical physics, and K-theory. Given a combinatorial Hopf algebra, the computation of its antipode gives combinatorial identities for the objects which built up the algebra. For this reason, finding the simplest expression for the antipode can prove extremely valuable.

In general the antipode is given in terms of satisfying certain commutative diagrams, but in the setting where the Hopf algebra is both graded and connected Takeuchi gave a formula which describes the map explicitly.

**Theorem 1.2.1 ([Tak71]).** A graded, connected $k$-bialgebra $H$ is a Hopf algebra, and it has a unique antipode $S$ whose formula is given by

$$ S = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^i \mu^{-i-1} \circ \pi \circ \Delta^{-i-1} $$

where $\mu^{-1} = \eta$, $\Delta^{-1} = \epsilon$, and $\pi : H \to H$ is the projection map defined by extending linearly the map

$$ \pi|_{H_\ell} = \begin{cases} 0 & \text{if } \ell = 0, \\ \text{id} & \text{if } \ell \geq 1, \end{cases} $$

where $H_\ell$ is the $\ell$th graded piece of $H$.

The only drawback to the formula above is that it can contain a large amount of cancellation. The goal in general is then to find refinements of this formula so that we can more easily compute the antipode of $H$. Ideally this leads to a cancellation-free formula. Much work has been done in this area for specific Hopf algebras. Cancellation-free formulas have been found for antipodes of combinatorial Hopf algebras associated to graphs, symmetric functions, shuffles, polynomials, simplicial complexes, and many more [HM12, LP07, Pat16, BS16, BHM16].

Our main result is a cancellation-free formula for computing the antipode of uniform matroids in the localization-contraction matroid Hopf algebra. This is a certain combinatorial Hopf algebra whose elements are matroids, and whose comultiplication map is defined in terms of localization and contraction (as in Section 1).

**Theorem 1.2.2 (Bucher–M. 2016 [BM16]).** The image of the uniform matroid $U_n^m$ under the antipode map $S$ is given by the following cancellation-free formula. Choose a total ordering $<$ on the ground set of the matroid. Then

$$ S(U_n^m) = \sum_{I,L} (-1)^{n-|L|+1} U_I^{m-|I|} \oplus U_L^{m-|I|}, $$

where $I, L$ ranges over all pairs of subsets of the ground set such that

- $I$ and $L$ are disjoint, and
- $|I| < m \leq |I| + |L|$, and
- if $|I| + |L| = m$ then $|L| = 1$ and the element in $L$ is the maximal element of $I \cup L$ with respect to $<$. \[ \]

Our proof of this theorem involves reinterpreting Theorem 1.2.1 for the localization-contraction matroid Hopf algebra; in this case, this reduces to a sum over certain ordered set partitions. To systematically remove cancellations from this sum, we introduce a sign-reversing involution on the collection of ordered set partitions. The idea of introducing a sign-reversing involution is ubiquitous throughout combinatorics, and it was used by Benedetti and Sagan to provide antipode formulas for various other combinatorial Hopf algebras [BS16].
Since our paper appeared on the arXiv, Aguiar and Ardila developed a completely general way to obtain a cancellation-free formula for all matroids in terms of the associated matroid polytope \cite{AA17}, thereby completing the problem of cancellation-free antipode formulas for the localization-contraction matroid Hopf algebra. Their methods also provide a unifying framework that applies to graphs, posets, set partitions, linear graphs, hypergraphs, simplicial complexes, building sets, and simple graphs.

2 Cluster Algebras

2.1 Computing upper cluster algebras

Cluster algebras are commutative unital domains generated by distinguished elements called cluster variables which are defined by a combinatorial process called mutation. Many notable spaces are equipped with cluster structures where certain regular functions play the role of cluster variables. For example, the coordinate ring of the space of \( m \times n \) matrices is naturally a cluster algebra, and each matrix minor is a cluster variable. In this way, identities among matrix minors are a special case of the theory.

Cluster algebras are generally defined in terms of an infinite generating set; however, the most important cluster algebras may be finitely-generated. For this reason, computing generating sets of cluster algebras and the relations between those generators is an interesting problem. The cluster algebra \( \mathcal{A} \) is the combinatorially defined object, but from a geometric perspective, there is a more natural algebra to consider: the upper cluster algebra \( \mathcal{U} \), which was introduced in \cite{BFZ05}. It is defined as an infinite intersection of Laurent polynomial rings, which makes \( \mathcal{U} \) difficult to work with in general, as it is hard to write down any (even infinite) generating set. Despite this hurdle, all known examples of upper cluster algebras enjoy many nice properties, such as normality and being log Calabi–Yau. Studying \( \mathcal{U} \) geometrically often gives information about the more intrinsic algebra \( \mathcal{A} \), since \( \mathcal{A} \subseteq \mathcal{U} \) by the Laurent phenomenon \cite{FZ02}.

One obstacle in the theory of cluster algebras has been an almost complete lack of examples in situations where \( \mathcal{A} \neq \mathcal{U} \). In a recent paper \cite{MM15}, G. Muller (University of Michigan) and I exploited the algebraic geometry of \( \mathcal{U} \) to obtain an algorithm for producing a presentation of \( \mathcal{U} \) (when \( \mathcal{A} \) is “totally coprime”—a mild technical condition) in terms of generators and relations.

Theorem 2.1.1 (M.–Muller 2015 \cite{MM15}). If \( \mathcal{A} \) is a totally coprime cluster algebra, then a finite-time algorithm for presenting \( \mathcal{U} \) exists whenever \( \mathcal{U} \) is finitely generated.

We used this technique to give presentations of several interesting upper cluster algebras where \( \mathcal{A} \neq \mathcal{U} \). Also, together with Mills, Muller, and Williams, I implemented the algorithm in the computer algebra system Sage (see \url{https://trac.sagemath.org/ticket/18800}).

![important_cluster_algebras_for_which_our_algorithm_gave_a_pres}(x)

Fig. 1: Important cluster algebras for which our algorithm gave a presentation of \( \mathcal{U} \).

Theorem 2.1.1 has been used by several other mathematicians in their research—I. Canakci, K. Lee, and R. Schiffler used our work to prove that \( \mathcal{A} = \mathcal{U} \) for the dreaded torus (the last quiver in Figure 1) \cite{CLS14}, and more recently, M. Gross, P. Hacking, S. Keel, and M. Kontsevich used our computations to make conjectures about the cluster variety for the once-punctured torus (the first quiver in Figure 1 with \( a = 2 \)) \cite{GHKK14}.
3 Representation Theory

3.1 Exceptional sequences and linear extensions

An exceptional sequence \((V_1, \ldots, V_n)\) of quiver representations is a sequence of representations obeying certain strong homological constraints. Exceptional sequences were introduced in [GR87] to study exceptional vector bundles on \(\mathbb{P}^2\). Since then, they have been shown to have many useful applications to several areas of mathematics. For example, they have applications in combinatorics because (complete) exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions [IT09, HK13]. Furthermore, they have been shown to be intimately connected to acyclic cluster algebras as their dimension vectors appear as rows of \(c\)-matrices [ST13].

Even though they are pervasive throughout mathematics, very little work has been done to give a concrete description of exceptional sequences. Previously, A. Garver (University of Minnesota) and I extended work of T. Araya [Ara13] by classifying exceptional sequences of representations of the linearly-ordered quiver of type \(A\) using noncrossing edge-labeled trees in a disk with boundary vertices [GM15].

To extend this classification to type \(A\) quivers with any orientation, Igusa, Garver, Ostroff, and I used a more general combinatorial model called a strand diagram. We defined a bijective map \(\Phi\), which takes an indecomposable representation to its corresponding strand. It turns out that all of the homological information in the definition of an exceptional sequence is stored in a noncrossing diagram of strands.

\[ X_{0,1} = k \leftarrow 0 \rightarrow 0 \xrightarrow{\Phi} + \xrightarrow{\Phi} \]
\[ X_{1,2} = 0 \leftarrow 0 \rightarrow k \xrightarrow{\Phi} + \xrightarrow{\Phi} \]
\[ X_{0,2} = k \leftarrow 0 \rightarrow 0 \xrightarrow{\Phi} + \xrightarrow{\Phi} \]

**Fig. 2:** The indecomposable representations of the type \(A_2\) quiver \(1 \leftarrow 2\) and their corresponding strands.

**Theorem 3.1.1** (Garver–Igusa–M.–Ostroff 2015 [GIMO15]). Let \(Q\) be a type \(A\) quiver and let \(U\) and \(V\) be two distinct indecomposable representations of \(Q\).

- a) The strands \(\Phi(U)\) and \(\Phi(V)\) intersect nontrivially if and only if neither \((U,V)\) nor \((V,U)\) are exceptional pairs.
- b) The strand \(\Phi(U)\) is clockwise from \(\Phi(V)\) if and only if \((U,V)\) is an exceptional pair and \((V,U)\) is not an exceptional pair.
- c) The strands \(\Phi(U)\) and \(\Phi(V)\) do not intersect at any of their endpoints and they do not intersect nontrivially if and only if \((U,V)\) and \((V,U)\) are both exceptional pairs.

**Theorem 3.1.1** has many immediate consequences. B. Keller proved that any two green-to-red sequences (sequences that are like MGS’s except that mutation at red vertices is allowed) applied to \(Q\) with principal coefficients produce isomorphic ice quivers [Kel13]. His proof hinges on deep representation-theoretic and geometric methods, but the statement itself is purely combinatorial. During the FPSAC’13 conference, B. Keller asked for a combinatorial proof—our theorem solves this problem. Another application of **Theorem 3.1.1** is that it allowed us to classify all \(c\)-matrices for type \(A\) cluster algebras in a completely combinatorial way using strand diagrams.

A complete exceptional collection is an unordered list of quiver representations which can be ordered to make a complete exceptional sequence. We have shown that the number of complete exceptional sequences arising from a given complete exceptional collection is equal to the number of linear extensions of a certain poset arising from the corresponding strand diagram. Counting the number of linear extensions of posets is a notoriously difficult problem in combinatorics; however, the class of posets arising from strand diagrams are very nice.

**Theorem 3.1.2** (Garver–Igusa–M.–Ostroff 2015 [GIMO15]). The class of posets arising from strand diagrams are exactly the posets \(\mathcal{P}\) that satisfy the following properties.
• each \( x \in \mathcal{P} \) has at most two covers and covers at most two elements,

• the underlying graph of the Hasse diagram of \( \mathcal{P} \) has no cycles,

• the Hasse diagram of \( \mathcal{P} \) is connected.

In particular, zig-zag posets are contained in this class, and their linear extensions are counted by a determinant formula. Together with Alejandro Morales, I have been working towards a formula for the number of linear extensions of all posets in this class.

**Problem 3.1.3** (M.–Morales). Count the number of complete exceptional sequences arising from a given complete exceptional collection by developing a method for counting the number of linear extensions of posets arising from strand diagrams.

### 3.2 Computing the Fourier–Sato transform combinatorially

In this section, I will describe a problem involving perverse sheaves on certain quiver varieties and discuss possible applications to representation theory of quantum loop algebras \( \mathcal{U}_q(Lg) \) and cluster algebras.

The Fourier–Sato transform is a geometric version of the Fourier transform for functions from analysis. If \( V \) is a complex vector space, then the Fourier–Sato transform is a certain functor \( T \) which gives an equivalence of categories between derived categories of conical sheaves

\[
T : D^{\text{b}}_{\text{con}}(V) \rightarrow D^{\text{b}}_{\text{con}}(V^*)
\]

where \( V^* \) is the dual vector space. The functor \( T \) has been used in various forms to much success for more than thirty years—it was used by Laumon in the mid-1980s to significantly shorten Deligne’s proof of the Weil conjectures (see [KW01] for a nice treatment), it gives an alternative construction of the Springer correspondence [HK84] [Bry86], and it plays a central role in the theory of character sheaves [Lus87] [Mir04]. Despite its usefulness, in practice \( T \) is difficult to compute explicitly. In fact, it has only been explicitly computed in few settings—some examples include the nilpotent cone \( \mathcal{N} \) of a semisimple Lie algebra \( g \) [AM15] and various stratified spaces of \( n \times n \) matrices [BG99].

P. Achar, M. Kulkarni, and I have started a program to compute \( T \) combinatorially. Let \( Q \) be the type \( A_n \) quiver \( \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \). Fix a dimension vector \( w \), and consider the space of representations of \( Q \) with this dimension vector. This is an affine space \( E_w \) which carries an action of a group \( G_w \), a product of general linear groups. This action gives a stratification of \( E_w \) into \( G_w \)-orbits, and we have the Fourier–Sato transform

\[
T : D^{\text{b}}_{G_w}(E_w) \rightarrow D^{\text{b}}_{G_w}(E_w^*)
\]

where \( w^* \) is the reverse of \( w \). In this setting, \( T \) is \( t \)-exact for the perverse \( t \)-structure, and it sends (simple) perverse sheaves to (simple) perverse sheaves.

To each \( G_w \)-orbit, we associate a triangular array of nonnegative integers satisfying certain conditions arising from dominant weights for \( GL_n \). These triangular arrays completely determine the \( G_w \)-orbits.

**Theorem 3.2.1** (Achar–Kulkarni–M. 2016). There is a bijection between \( G_w \)-orbits in \( E_w \) and a certain set \( P_w \) of triangular arrays of nonnegative integers.

The combinatorial set \( P_w \) carries a wealth of information about the geometry and representation theory of \( E_w \)—the dimension of \( G_w \)-orbits and the partial order on them, as well as whether a representation is injective or projective, can be read off the diagram. Most notably, it (conjecturally) allows for a combinatorial computation of \( T \).

**Conjecture 3.2.2** (Achar–Kulkarni–M. 2016, proof in progress). There exists a combinatorial algorithm \( T : P_w \rightarrow P_w^* \) that computes \( T \) for every simple perverse sheaf; i.e., such that \( T(\text{IC}(O_\lambda)) = \text{IC}(O_{T(\lambda)}) \), where \( \text{IC}(O_\lambda) \) is the simple perverse sheaf supported on the closure \( \overline{O_\lambda} \) of the \( G_w \)-orbit \( O_\lambda \) given by \( \lambda \in P_w \).
We give a few examples of $T$ below for the $A_3$ quiver with dimension vector $w = (3,3,3)$. Note that the first example is the combinatorial version of the geometric fact that $T$ takes the unique zero-dimensional orbit to the unique dense orbit.

$$
\begin{array}{ccc}
T\left(\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}\right) & = & \left(\begin{array}{c}
0 \\
1 \\
0 \\
\end{array}\right) \\
T\left(\begin{array}{c}
2 \\
3 \\
1 \\
\end{array}\right) & = & \left(\begin{array}{c}
0 \\
2 \\
2 \\
\end{array}\right) \\
T\left(\begin{array}{c}
3 \\
2 \\
1 \\
\end{array}\right) & = & \left(\begin{array}{c}
2 \\
3 \\
0 \\
\end{array}\right)
\end{array}
$$

Recently, Nakajima used the Fourier–Sato transform on his graded quiver varieties to prove a character formula for representations of quantum loop algebras $\mathcal{U}_q(Lg)$, as well as to give a monoidal categorification of certain cluster algebras [Nak11]. In his proof, he computes the Fourier–Sato transform of only a single object. Conjecture 3.2.2 allows for the computation of $T$ for every object, at least for the quiver $Q$.

**Problem 3.2.3.** Get information about representations of $\mathcal{U}_q(Lg)$ by using $T$, our combinatorial description of $T$. What does our explicit computation of $T$ say about the cluster algebras involved?

This problem is particularly interesting to me as cluster algebras have been another important research interest of mine—I have results about their structure theory (see Section 2.1) and their relationship with quiver representations (see Section 3.1). Solving this problem would finally allow me to blend geometric representation theory, quivers, and cluster algebras (several of my research interests).

### 3.3 Derived geometric Satake equivalence, Springer correspondence, and small representations

Let $G$ be a semisimple complex algebraic group and $T$ be a maximal torus. The Weyl group $W$ acts on the zero weight space $V^T$ of any representation $V$ of $G$, giving a functor $\Phi_G : \text{Rep}(G) \to \text{Rep}(W)$.

Achar, Henderson, and Riche recently constructed a geometric lift of this functor [AHI13 AHR15]. Its construction begins with a result of Lusztig [Lus81] that for $GL_n(C)$ the nilpotent cone $\mathcal{N}$ (the variety of nilpotent $n \times n$ matrices) can be embedded in the affine Grassmannian $\text{Gr}_{GL_n(C)}$, an infinite-dimensional analog of the Grassmannian of $k$ planes in $n$ space. To generalize this result to other groups, one needs to restrict to a certain finite-dimensional closed subvariety $\text{Gr}^{sm}$ of $\text{Gr}$. There is a certain open subvariety $\mathcal{M} \subset \text{Gr}^{sm}$ and a finite map $\pi : \mathcal{M} \to \mathcal{N}$, which can be viewed as a generalization of Lusztig’s embedding for other groups. The map $\pi$ gives rise to a functor $\Psi_G : \text{Perv}_{\tilde{G}(\mathcal{D})}(\mathcal{G}) \to \text{Perv}_{\tilde{G}(\mathcal{N})}$.

To explain in what sense $\Psi_G$ is a geometric lift of $\Phi_G$, we invoke two major theorems in geometric representation theory: the geometric Satake equivalence and the Springer correspondence.

(a) There is an equivalence of categories $S : \text{Perv}_{\tilde{G}(\mathcal{D})}(\text{Gr}_{\tilde{G}}) \to \text{Rep}(G)$, where $\text{Perv}_{\tilde{G}(\mathcal{D})}(\text{Gr}_{\tilde{G}})$ is the category of $\tilde{G}(\mathcal{D})$-equivariant perverse sheaves on the affine Grassmannian $\text{Gr}_{\tilde{G}} := \tilde{G}((t)) \backslash \tilde{G}(\mathcal{D})$, $\tilde{G}$ is the Langlands dual group, $\mathcal{D} = \mathbb{C}[[t]]$, and $\mathcal{S} = \mathbb{C}((t))$. [Lus83 Gin95 MV07].

(b) There is a functor $S : \text{Perv}_G(\mathcal{N}) \to \text{Rep}(W)$, where $\text{Perv}_G(\mathcal{N})$ is the category of $G$-equivariant perverse sheaves on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ [Spr76 Lus81 BM81].

**Theorem 3.3.1** ([AHI13 AHR15]). The functor $\Psi_G$ is a geometric lift of $\Phi_G$; i.e., there exists a commutative diagram:

$$
\begin{array}{ccc}
\text{Perv}_{\tilde{G}(\mathcal{D})}(\text{Gr}_{\tilde{G}}^{sm}) & \xrightarrow{S_{sm}} & \text{Rep}(G)^{sm} \\
\downarrow{\Phi_G} & & \downarrow{\Phi_G} \\
\text{Perv}_G(\mathcal{N}) & \xrightarrow{S} & \text{Rep}(W)
\end{array}
$$

Parallel to this development, derived versions of (a) and (b) were developed—Bezrukavnikov and Finkelberg gave an equivalence $\text{der}S : \mathcal{D}^b_{\tilde{G}(\mathcal{D})}(\text{Gr}_{\tilde{G}}) \to \mathcal{D}^b_{\mathcal{G}}\text{Coh}^{G \times G_m}(\mathfrak{g}^*)$ [BF08], and Rider established the equivalence $\text{der}S : \mathcal{D}^b_{\tilde{G}(\mathcal{D})}(\mathcal{N}_G) \to \mathcal{D}^b_{\mathcal{G}}\text{Coh}^{W \times G_m}(\mathfrak{h}^*)$ [Rid13]. Here, $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are both affine varieties, so the categories on the right-hand side are graded modules over their coordinate rings. The word “mix” refers to a geometric analog of grading. My dissertation work involved producing a derived analog of $\Phi_G$ and showing that its derived lift is $\Psi_G$. 


Theorem 3.3.2 (M. 2016 [Mat16]). There exists a commutative diagram of derived categories:

\[
\begin{array}{ccc}
\mathcal{D}^b_{G(O)}(\text{Gr}^\text{sm}_G) & \xrightarrow{\Psi_G} & \mathcal{D}^b_{\text{Coh}(G)}(\mathfrak{g}^*)_{\text{sm}} \\
\Psi_G & & \downarrow \text{der}\Phi_G \\
\mathcal{D}^b_{G,Spr}(\mathcal{N}_G) & \xrightarrow{\sim}_{\text{der}\mathbb{S}} & \mathcal{D}^b_{\text{Coh}(W\times G^m)}(\mathfrak{h}^*)
\end{array}
\]

The solution to this problem involves two major steps.

1. Establish the result for all groups \(G\) of semisimple rank 1.

2. Show that each of the functors involved commutes with restriction to such a group.

Over fields \(F\) of positive characteristic, representation theory is much more difficult—for example, the categories involved are not semisimple in general. In his thesis, Mautner proved that for \(G = \text{GL}_n(F)\) the category \(\text{Perv}_G(\mathcal{N})\) is equivalent to the category of finitely-generated modules for a certain Schur algebra [Man10]. Very recently, tremendous progress has been made toward producing analogs of Springer theory in positive characteristic [AHJR16, AHJR14, AHJR15]. It is motivating to consider a positive characteristic version of Theorem 3.3.2 for its possible applications to positive characteristic Springer theory and representations of Schur algebras.

Problem 3.3.3. Develop an analog of Theorem 3.3.2 for sheaves in positive characteristic.

References


