1. (10 points) Let $V$ be an $n$-dimensional vector space over the field $\mathbb{R}$ and let $T : V \to \mathbb{R}^2$ be a linear transformation from $V$ to $\mathbb{R}^2$. Prove that if $T$ is not the zero transformation and $T$ is not onto, then $\dim(\text{null}(T)) = n - 1$, where $\text{null}(T) := \{ v \in V : T(v) = 0 \}$.

**Answer:** (Compare with problem 5 page 108 in the text). The subspace $T(V)$ of $\mathbb{R}^2$ has dimension $0 < \dim(T(V)) < 2$, since $T$ is not the zero transformation and $T$ is not onto. Hence $\dim(T(V)) = 1$. The Fundamental Theorem of Linear Algebra states, that

$$
\dim(\text{null}(T)) + \dim(T(V)) = \dim(V).
$$

Hence, $\dim(\text{null}(T)) = \dim(V) - \dim(T(V)) = n - 1$.

2. (10 points) Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, satisfying $T(1,1,1) = (1,2)$, $T(1,2,1) = (1,1)$, and $T(2,1,2) = (2,1)$. Justify your answer!

**Answer:** (Compare with Problem 11 page 107 in the text). First check if there are any non-trivial linear relations among the three vectors in $\mathbb{R}^3$, by forming the $3 \times 3$ matrix $A$, with these vectors as columns, so that the coefficient vector of any linear relation among them is a solution of $Ax = 0$. Row reducing, we get:

$$
A = \begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}
\sim \cdots \sim
\begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}.
$$

The general solution is

$$
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = x_3
\begin{pmatrix}
-3 \\
1 \\
1
\end{pmatrix}.
$$

We get the linear relation:

$$
-3(1,1,1) + (1,2,1) + (2,1,2) = (0,0,0).
$$

If $T$ exists and we apply $T$ to both sides of the above relation, we get

$$
-3(1,2) + (1,1) + (2,1) = (0,0).
$$

The left hand side is $(0, -4)$, so such a $T$ does not exist.

3. (20 points) Let $V$ be the vector space of all polynomial functions

$$
f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3
$$

of degree $\leq 3$ with real coefficients $c_i$, and $T : V \to V$ the linear transformation

$$
T(f) = (x + 1) \frac{\partial f}{\partial x} - f
$$

sending $f$ to $(x + 1)$ times its derivative minus $f$ itself.

(a) (10 points) Find the matrix $[T]_\beta$ in the basis $\beta = \{1, x, x^2, x^3\}$ of $V$.

**Answer:** (Compare with Problem 3 page 108 in the text).

$$
[T]_\beta = ([T(1)]_\beta[T(x)]_\beta[T(x^2)]_\beta[T(x^3)]_\beta)
= \left( [-1]_\beta [1]_\beta [2x + x^2]_\beta [3x^2 + 2x^3]_\beta \right) =
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}
$$
5. (20 points) Let \( \langle \rangle \) be an inner product.
(a) (8 points) Use the Gram-Schmidt process, and the above basis \( \{v_1, v_2, v_3\} \) of \( \mathbb{R}^3 \), to find an orthonormal basis \( \{u_1, u_2, u_3\} \) of \( \mathbb{R}^3 \), such that \( \text{span}\{v_1, \ldots, v_r\} = \text{span}\{u_1, \ldots, u_r\} \), for \( 1 \leq r \leq 3 \).

Answer: \( (\text{Compare with problem 1 page 129}) \) \( u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) \).
\( v_2 - (v_2, u_1)u_1 = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = \frac{1}{2}(1, -1, 2) \). Normalize to get

(b) (3 points) Find a basis for the null space \( \text{null}(T) := \{ f : T(f) = 0 \} \). Justify your answer!

Answer: The row reduced echelon form of \( [T]_{\beta} \) is \( B := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

The general solution of \( Bx = 0 \) is \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \), which is the subspace spanned by the coordinate vector \( [1+x]_{\beta} \). Hence, \( \text{null}(T) \) is spanned by \( 1 + x \).

(c) (2 points) Determine the rank of \( T \).

Answer: The rank of \( T \) is equal to the rank of its matrix \( [T]_{\beta} \), which is equal to the rank of its reduced echelon form \( B \), which is 3.

(d) (5 points) Find a basis for the image \( T(V) \) of \( T \) (consisting of polynomials!!!).

Answer: The pivot columns of the reduced echelon form \( B \) are the first, third, and fourth. Hence, the first, third, and fourth columns of the matrix \( [T]_{\beta} \) are a basis for the column space of \( [T]_{\beta} \). Thus, \( T(1), T(x^2), T(x^3) \) form a basis for \( T(V) \). We calculated above that these three vectors are \( -1, 2x + x^2, 3x^2 + 2x^3 \). There are many other correct answers.

4. (20 points) Let \( V \) be a finite dimensional vector space over the real numbers, with an inner product. Recall that a linear transformation \( T : V \to V \) is called an orthogonal transformation, if it preserves length, i.e., \( \|T(v)\| = \|v\| \), for all \( v \in V \).

(a) (10 points) Prove that the product \( TS \), of two orthogonal transformations \( T \) and \( S \), is an orthogonal transformation.

Answer: Compare with problem 7 page 129. \( TS(v) = T(S(v)) \), so
\( \|TS(v)\| = \|T(S(v))\| = \|S(v)\| = \|v\| \), where the second equality is due to the assumption, that \( T \) is orthogonal, and the last equality is due to \( S \) being orthogonal.

(b) (10 points) Let \( T \) be an orthogonal transformation of \( V \). Show that \( \det(T) \) is equal to \( 1 \) or \( -1 \).

Answer: Compare with problem 5 page 150. Let \( \beta = \{u_1, \ldots, u_n\} \) be an orthonormal basis for \( V \). The matrix \( A := [T]_{\beta} \) satisfies \( (t^*A)A = I \) (the transpose of \( A \) is equal to the inverse of \( A \)), by Theorem 15.11 page 127. Recall that \( \det(A) = \det(t^*A) \). Hence,
\[ 1 = \det(I) = \det((t^*A)A) = \det(t^*A) \det(A) = (\det(A))^2. \]
Thus, \( \det(T) = \det(A) = \pm 1 \).

5. (20 points) Let \( v_1 = (1, 1, 0) \), \( v_2 = (1, 0, 1) \), and \( v_3 = (1, 1, 1) \).

(a) (8 points) Use the Gram-Schmidt process, and the above basis \( \{v_1, v_2, v_3\} \) of \( \mathbb{R}^3 \), to find an orthonormal basis \( \{u_1, u_2, u_3\} \) of \( \mathbb{R}^3 \), such that \( \text{span}\{v_1, \ldots, v_r\} = \text{span}\{u_1, \ldots, u_r\} \), for \( 1 \leq r \leq 3 \).

Answer: \( (\text{Compare with problem 1 page 129}) \) \( u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) \).
\( v_2 - (v_2, u_1)u_1 = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = \frac{1}{2}(1, -1, 2) \). Normalize to get
\[ u_2 = \frac{(1,1,2)}{\| (1,1,2) \|} = \frac{1}{\sqrt{6}} (1, -1, 2). \]
\[ v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2 = (1, 1, 1) - (1, 1, 0) - \frac{1}{3}(1, -1, 2) = \frac{1}{3}(-1, 1, 1). \]
\[ u_3 = \frac{(-1,1,1)}{\|(-1,1,1)\|} = \frac{1}{\sqrt{3}} (-1, 1, 1). \]

(b) (4 points) State the definition of an orthonormal basis, and check that the basis you found in part 5a is orthonormal.

**Answer:** An orthonormal basis for an n-dimensional vector space is a set of n vector \( \{u_1, \ldots, u_n\} \), satisfying \( (u_i, u_i) = 1 \), for all \( i \), and \( (u_i, u_j) = 0 \), if \( i \neq j \). Two points were given for the check of your answer in part 5a.

(c) (4 points) Find the distance from the vector \( v_3 \) to the plane spanned by \( \{v_1, v_2\} \). (these vectors are given at the beginning of problem 5).

**Answer:** Compare with problem 13 page 131. The plane \( P \) spanned by \( \{v_1, v_2\} \) is also spanned by \( \{u_1, u_2\} \). We need an orthonormal basis for \( P \) in order to compute the projection \( \hat{v}_3 := (v_3, u_1)u_1 + (v_3, u_2)u_2 \) of \( v_3 \) to \( P \)!!! We get
\[ \hat{v}_3 := (v_3, u_1)u_1 + (v_3, u_2)u_2 = (1, 1, 0) + \frac{1}{3}(1, -1, 2) = \frac{2}{3}(2, 1, 1). \]
But, in fact, we need the difference \( v_3 - \hat{v}_3 \), which was already calculated in part 5a. The distance is \( \| v_3 - \hat{v}_3 \| = \frac{1}{\sqrt{3}}(-1, 1, 1) = \frac{1}{\sqrt{3}} \).

(d) (4 points) Explain how to read, from the orthonormal basis you found in part 5a, without any further computations, the equation of the plane spanned by \( \{v_1, v_2\} \).

**Answer:** Compare with part (a) of problem 11 page 130. The vector \( u_3 \) is orthogonal to the plane \( P \) spanned by \( \{u_1, u_2\} \), which is equal to the plane spanned by \( \{v_1, v_2\} \). Hence the plane \( P \) is equal to \( \{v : (v, u_3) = 0\} \). Using dot product, it becomes
\[ P = \{(x_1, x_2, x_3) : \frac{1}{\sqrt{3}}(-x_1 + x_2 + x_3) = 0\}, \]
or simply \(-x_1 + x_2 + x_3 = 0\).

6. (20 points) Let \( V \) be an n-dimensional vector space with an inner product and \( u \) a unit vector in \( V \) (so that \( (u, u) = 1 \)). Let \( u^\perp \) be the subspace \( \{v \in V : (v, u) = 0\} \), orthogonal to \( u \). Recall that the reflection \( R_u : V \to V \), of \( V \) with respect to \( u^\perp \), is given by
\[ R_u(v) = v - 2(v, u)u. \]

(a) (8 points) Prove that \( R_u \) is a linear transformation (it is also easy to show that \( R_u \) is an orthogonal transformation, but you are not asked to show it).

**Answer:** Check the two properties in the definition of a linear transformation: 1) For every two vectors \( v_1, v_2 \in V \) we have
\[ R_u(v_1 + v_2) = v_1 + v_2 - 2(v_1 + v_2, u)u = v_1 + v_2 - 2(v_1, u)u - 2(v_2, u)u \]
\[ = [v_1 - 2(v, u)u] + [v_2 - 2(v_2, u)u] = R_u(v_1) + R_u(v_2). \]
2) For every \( \lambda \in \mathbb{R} \) and every \( v \in V \), we have
\[ R_u(\lambda v) = \lambda v - 2(\lambda v, u)u = \lambda v - 2\lambda (v, u)u = \lambda [v - 2(v, u)u] = \lambda R_u(v). \]
(b) (4 points) Let $u_1$ and $u_2$ be two unit vectors in $V$. Show that if $(u_1, u_2) = 0$, then $R_{u_1} R_{u_2} = R_{u_2} R_{u_1}$. In other words, the two reflections commute, if the two unit vectors are orthogonal.

**Answer:** We need to prove the equality $R_{u_1} R_{u_2}(v) = R_{u_2} R_{u_1}(v)$ for every vector $v$ in $V$.

$$
R_{u_1} R_{u_2}(v) = R_{u_1}(R_{u_2}(v)) = R_{u_1}(v - 2(v, u_2)u_2) =
$$

$$
= [v - 2(v, u_2)u_2] - 2([v - 2(v, u_2)u_2], u_1) u_1
$$

$$
= v - 2(v, u_2)u_2 - 2(v, u_1)u_1 + 4(v, u_2)(u_2, u_1) u_1
$$

$$
= v - 2(v, u_2)u_2 - 2(v, u_1)u_1.
$$

The last equality uses the vanishing $(u_2, u_1) = 0$. Now the last term we got is symmetric in $u_1$ and $u_2$ and so is equal also to $R_{u_2} R_{u_1}(v)$.

(c) (8 points) Let $V = \mathbb{R}^2$, with the standard inner product (the dot product), and set $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Find the matrix $[R_u]_\beta$, of the reflection $R_u$, with respect to the basis $\beta = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. Justify your answer!

**Answer:** Compare with part b.iii of the additional problem to section 18. (Parts b.i and b.ii of that problem were added after the exam for future semesters). Set $u_1 := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $u_2 := (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then

$R_u(u_1) = u_1 - 2(u_1, u)u = u_1 - 2(u_1, u_1)u_1 = -u_1$ and

$R_u(u_2) = u_2 - 2(u_2, u)u = u_2 - 0 = u_2$. We get

$$
[R_u]_\beta = ([R_u(u_1)]_\beta[R_u(u_2)]_\beta) = ([u_1]_\beta[u_2]_\beta) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$