Additional problems for section 25

1. Let $T$ be a linear transformation on a vector space $V$ over the complex number field $\mathbb{C}$ with a basis $\{u, v, w\}$, such that

$$T(u) = u - v$$
$$T(v) = u + 3v$$
$$T(w) = -u - 4v - w.$$ 

(a) Find the elementary divisors of $T$.
(b) Find the Jordan canonical form of $T$.
(c) Find a basis $v_1, v_2, v_3$ of $V$, such that the matrix of $T$ in this basis is the Jordan canonical form of $T$.

2. Continuation of Problem 8 from section 25 page 226

(a) Let $A$ be a matrix, whose elementary divisors are

$$\{p_1(x)^{e_{1,1}}, \ldots, p_1(x)^{e_{1,k_1}}, p_2(x)^{e_{2,1}}, \ldots, p_2(x)^{e_{2,k_2}}, \ldots, p_r(x)^{e_{r,1}}, \ldots, p_r(x)^{e_{r,k_r}}\},$$

where $p_i(x)$, $1 \leq i \leq r$ are distinct prime polynomials, and $e_{i,j}$ are positive integers. Prove that the minimal polynomial of $A$ is

$$m(x) = p_1(x)^{e_1} \cdot p_2(x)^{e_2} \cdots p_r(x)^{e_r}, \tag{1}$$

where $e_i = \max\{e_{i,1}, \ldots, e_{i,k_i}\}$. Hint: Let $f(x)$ be the polynomial on the right hand side of (1). Prove that $f(A) = 0$. Prove also that $p_i(x)^{e_i}$ divides the minimal polynomial $m(x)$.

(b) Conclude, that if the minimal polynomial $m(x)$ of $A$ is equal to the characteristic polynomial $h(x)$, then the elementary divisors of $A$ are determined by $h(x)$. (See the Unique Factorization Theorem 20.18 page 171 in the text).

(c) Let $A$ and $B$ be two $2 \times 2$ matrices with entries in a field $F$. Show that $A$ and $B$ are similar, if and only if they have the same minimal polynomial.

(d) Let $A$ and $B$ be two $3 \times 3$ matrices with entries in a field $F$. Show that $A$ and $B$ are similar, if and only if they have the same characteristic polynomial $h(x)$ and the same minimal polynomial $m(x)$.

(e) Give an example of two $4 \times 4$ matrices $A$ and $B$, which are not similar, but which have the same characteristic polynomial $h(x)$ and the same minimal polynomial $m(x)$. 

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