Additional problems for section 25

1. Let T be a linear transformation on a vector space V over the complex number field $\mathbb{C}$ with a basis $\{u, v, w\}$, such that

$$\begin{aligned} T(u) &= u - v \\ T(v) &= u + 3v \\ T(w) &= -u - 4v - w. \end{aligned}$$

   (a) Find the elementary divisors of T.

   (b) Find the Jordan canonical form of T.

   (c) Find a basis v_1, v_2, v_3, of V, such that the matrix of T in this basis is the Jordan canonical form of T.

2. Continuation of Problem 8 from section 25 page 226

   (a) Let A be a matrix, whose elementary divisors are

$$\{p_1(x)^{e_{1,1}}, \ldots, p_1(x)^{e_{1,k_1}}; p_2(x)^{e_{2,1}}, \ldots, p_2(x)^{e_{2,k_2}}, \ldots, p_r(x)^{e_{r,1}}, \ldots, p_r(x)^{e_{r,k_r}}\},$$

   where $p_i(x)$, $1 \le i \le r$ are distinct prime polynomials, and $e_{i,j}$ are positive integers. Prove that the minimal polynomial of A is

$$m(x) = p_1(x)^{e_1} \cdot p_2(x)^{e_2} \cdots p_r(x)^{e_r}, \tag{1}$$

   where $e_i = \max\{e_{i,1}, \ldots, e_{i,k_i}\}$. *Hint: Let $f(x)$ be the polynomial on the right hand side of (1). Prove that $f(A) = 0$. Prove also that $p_i(x)^{e_i}$ divides the minimal polynomial $m(x)$.*

   (b) Conclude, that if the minimal polynomial $m(x)$ of A is equal to the characteristic polynomial $h(x)$, then the elementary divisors of A are determined by $h(x)$. (See the Unique Factorization Theorem 20.18 page 171 in the text).

   (c) Let A and B be two 2×2 matrices with entries in a field F. Show that A and B are similar, if and only if they have the same minimal polynomial.

   (d) Let A and B be two 3×3 matrices with entries in a field F. Show that A and B are similar, if and only if they have the same characteristic polynomial $h(x)$ and the same minimal polynomial $m(x)$.

   (e) Give an example of two 4×4 matrices A and B, which are **not** similar, but which have the same characteristic polynomial $h(x)$ and the same minimal polynomial $m(x)$.