Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by multiplication by a 3×3 matrix A. We endow $\mathbb{R}^3$ with the dot product.

1. Show that T is a rotation of $\mathbb{R}^3$ about some line through the origin (the axis of rotation), if and only if A satisfies the following two conditions:

   (i) $A^T A = I$ (in other words, the transpose of A is equal to the inverse of A) and

   (ii) $\det(A) = 1$.

2. (a) Show that the following two are matrices of rotations (check conditions (i) and (ii) above):

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

   (b) Consider a rotation given by some matrix A. Explain why any vector in the axis line of the rotation must be an eigenvector of A with eigenvalue 1.

   (c) Find a non-zero vector spanning the axis line of each of the two rotations given by the matrices A and B above. *Note: For the axis line of B, you will probably need to use the identity* $(\sqrt{2}+1)(\sqrt{2}-1) = 1$.

   (d) Find an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$, such that $P^{-1}BP$ is in the normal form of Theorem 30.5 page 270, where $P = (u_1 u_2 u_3)$ is the (orthogonal) change of basis matrix.