Solve 6 out of the following 7 problems.
Show all your work and justify all your answers!!!
iv. Use the Jordan decomposition of $A$ to calculate the entries of $A^k$, as functions of $k$, for all positive integers $k$.

5. (17 points) Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ with an inner product and $u$ a unit vector in $V$. Recall, that the reflection $R$ of $V$, with respect to the subspace $u^\perp$ orthogonal to $u$, is given by

$$R(v) = v - 2(v, u)u.$$  

(a) Show that $R^2 = 1$.

(b) Find the minimal polynomial of $R$. Justify your answer.

(c) Show that $R$ is diagonalizable.

(d) Show that the $-1$ eigenspace of $R$ is spanned by $u$.

(e) Find the characteristic polynomial of $R$. Justify all your answers!

(f) Calculate the trace $\text{tr}(R)$.

6. (17 points) Let $V$ be a finite dimensional vector space over a field $F$, and $T : V \to V$ a linear transformation.

(a) Let $v \in V$ be an eigenvector of $T$ with eigenvalue $\lambda$, and $g(x) = c_n x^n + \cdots + c_0$ a polynomial in $F[x]$. Show that $v$ is an eigenvector of $g(T)$ and find its eigenvalue.

(b) Use part 6a to show, that every root of the characteristic polynomial $h(x)$ of $T$ is also a root of the minimal polynomial $m(x)$ of $T$ (without using the Cayley-Hamilton Theorem).

7. (17 points) Let $V$ be a four dimensional vector space over $\mathbb{C}$. Assume that the characteristic polynomial of $T$ is $(x - \lambda_1)^2(x - \lambda_2)^2$, and $\lambda_1 \neq \lambda_2$.

(a) What are all the possible minimal polynomials $m(x)$ of $T$ (with leading coefficient 1)? Justify your answer!

(b) Assume that the minimal polynomial of $T$ is $m(x) = (x - \lambda_1)^{e_1}(x - \lambda_2)^{e_2}$, set $V_i := \text{null}[(T - \lambda_i 1)^{e_i}]$, where $1$ is the identity transformation, and let $T_i \in L(V_i, V_i)$ be the restriction of $T$ to $V_i$. Use the Primary Decomposition Theorem to show, that the minimal polynomial of $T_i$ is $(x - \lambda_i)^{e_i}$. Hint: Show first that the minimal polynomial $m_i(x)$ of $T_i$ divides $m(x)$ and the product $g(x) := m_1(x)m_2(x)$ satisfies $g(T) = 0$.

(c) Assume that the minimal polynomial of $T$ is $(x - \lambda_1)^2(x - \lambda_2)$. Calculate the dimensions of the null spaces of $T - \lambda_1 1$, $(T - \lambda_1 1)^2$, $T - \lambda_2 1$, and $(T - \lambda_2 1)^2$. Carefully explain how your answer follows from the Primary Decomposition Theorem and the Triangular Form Theorem.