Chiriac's Practive Midterm 1
Solution

1) (a) The equation $44x + 17y = c$ has a solution (x, y) with $x, y \in \mathbb{Z}$, for every $c \in \mathbb{Z}$, since $\gcd(\underbrace{44}_{2^2 \cdot 11}, 17) = \underline{1}$

(Theorem 5.1.2).

(b) $44x + 17y \overset{(*)}{=} 100$

First solve $44x + 17y = 1$

$44x_i + 17y_i = r_i$

x_i	y_i	r_i	q_i
1	0	44	—
0	1	17	—
1	-2	10	2
-1	3	7	1
2	-5	3	1
-5	13	1	2
		0	

gcd(44, 17)

$44 \cdot (-5) + 17 \cdot 13 = 1$

A particular solution to $(*)$ is

$(x_0, y_0) = 100(-5, 13) = (-500, 1300)$

The general solution of $(*)$ is:

$\{(x, y) = (-500 - 17k, 1300 + 44k) ; k \in \mathbb{Z}\}$.

2) Solve (1) $t \equiv 1 \pmod 3$
(2) $t \equiv 2 \pmod 5$
(3) $t \equiv 3 \pmod 7$.

Step 1; Find $t_0 \in \mathbb{Z}$, such that the general solution of (1) and (2) is also the general solution of the congruence

$$(1 \,\&\, 2) \quad t \equiv t_0 \pmod{\underbrace{3\cdot 5}_{15}}$$

Solve the Diophantine eq

$$3x + 5y = \underbrace{2-1}_{1}$$

$(x_0, y_0) = (-3, 2)$ is a solution.

Set $t_0 = \underbrace{3\underbrace{x_0}_{-3} + 1}_{-8} = \underbrace{-5y_0 + 2}_{-8}$. Then t_0 solves (1) and (2)

Then the general solution of (1) and (2) is $\{t : t \equiv -8 \pmod{15}\}$, by the C.R.T.

Step 2; Solve the two congruences

$$(1 \,\&\, 2) \quad t \equiv -8 \pmod{15}$$
$$(3) \quad t \equiv 3 \pmod 7$$

Find a sol'n to the Diophantine Eq

$$15x + 7y \overset{(*)}{=} 3 - (-8) = 11$$

A sol'n to $15x + 7y = \gcd(15, 7) = 1$ is $(x, y) = (1, -2)$. So $(x_1, y_1) = 11(1, -2) = (11, -22)$ is a sol'n of $(*)$. Set $t_1 = \underbrace{15x_1 + (-8)}_{157} = \underbrace{-7y_1 + 3}_{157}$.

The general sol'n of (1), (2), and (3) is

$$\{t : t \equiv 157 \pmod{3\cdot 5\cdot 7}\}.$$

3 (a) Let the two amounts be

$$a = a_{m-1} a_{m-2} \cdots a_1 a_0$$

$$b = b_{m-1} b_{m-2} \quad b_1 b_0$$

where $\underbrace{b_i}_{0 \le i \ne j \le m-1} = a_j$, $b_j = a_i$ and $b_r = a_r$ if $r \in \{0, \ldots, m-1\} \setminus \{i, j\}$.

Assume that $i < j$.

Then $a - b = (a_j - b_j) 10^j + (a_i - b_i) 10^i =$

$= (a_j - a_i) 10^j + (a_i - a_j) 10^i =$

$= (a_j - a_i) \cdot 10^i \cdot \left[10^{j-i} - 1 \right]$.

Now, $10^{(j-i)} - 1 \equiv 1^{(j-i)} - 1 \equiv 1 - 1 \equiv 0 \pmod 9$.

Hence, $9 \mid a - b$.

Q.E.D

3 (b). We have proven in class that a natural number

$$a = a_{n-1} a_{n-2} \cdots a_1 a_0$$

is divisible by 11, if the alternating sum S of its digits is divisible by 11, where $S = a_0 - a_1 + \cdots + (-1)^{n-1} a_{n-1}$.

If n is even, and a is a palindrome, then $a_i = a_{n-i-1}$, for $0 \leq i \leq n-1$, but

$$(-1)^i a_i + \underbrace{(-1)^{n-i-1}}_{= (-1)^{i+1}} \underbrace{a_{n-i-1}}_{= a_i} = a_i (-1)^i (1-1) = 0.$$

Hence, $S = 0$, so $11 \mid S$, so $11 \mid a$.

Q.E.D.

Let $a, b, c \in \mathbb{N}$,

4) (a) Prove that $\gcd(a, b, c) = \gcd(\gcd(a,b), c)$.

This is homework problem 17 from Sec 5.3 whose solution is on our website

(b) Prove that the Diophantine Eq

$$ax + by + cz = 1$$

has a solution, if and only if $\gcd(a,b,c) = 1$.

Proof: Set $d := \gcd(a,b)$. Then the Diophantine Equation

$$(1) \qquad ax + by = d$$

has a solution (x_0, y_0), by the Extended Euclidean Algorithm Theorem,

Furthermore,

$$\underbrace{1}_{\boxed{\text{by assumption}}} = \gcd(a,b,c) \underset{\boxed{\text{by Part (a)}}}{=} \gcd(d, c).$$

Hence, the Diophantine Equation

$$dw + cz = 1$$

has a solution (w_1, z_1) with $w_1, z_1 \in \mathbb{Z}$.

Thus, set $x_1 = w_1 x_0$, $y_1 = w_1 y_0$. Then

$$ax_1 + by_1 + cz_1 = a w_1 x_0 + b w_1 y_0 + cz_1 =$$

$$= (ax_0 + by_0) w_1 + cz_1 \underset{\text{since } (x_0, y_0) \text{ solves (1)}}{=} dw_1 + cz_1 = 1.$$

Q.E.D.

5 (a) How many positive integers less than 101 have an odd number of positive divisors?

If $n = p_1^{e_1} \cdots p_r^{e_r}$ with p_j prime, $p_i \neq p_j$ if $i \neq j$, and $e_j \in \mathbb{N}$ for $1 \leq j, i \leq r$, then the number N of divisors of n is $(e_1+1)(e_2+1)\cdots(e_r+1)$.

For N to be odd, each exponent e_i must be even, $1 \leq i \leq r$. In particular, $e_j \geq 2$, for all j. So $p_i < \sqrt{101}$ for all i. So $p_i \in \{2, 3, 5, 7\}$.

$n = 2^{b_1} 3^{b_2} 5^{b_3} 7^{b_4}$, where $0 \leq b_i$ is even.

$0 \leq b_4 \leq 2 \qquad b_4 \in \{0, 2\}$

$0 \leq b_3 \leq 2 \qquad b_3 \in \{0, 2\}$

$0 \leq b_2 \leq 4 \qquad b_2 \in \{0, 2, 4\}$

$0 \leq b_1 \leq 6 \qquad b_1 \in \{0, 2, 4, 6\}$

1 { • If $b_4 = 2$, then $n = 7^2 = 49$.

Assume $\beta_4 = 0$.

2 If $\beta_3 = 2$, then $m = 5^2$, or $m = \overbrace{2^2 \cdot 5^2}^{100}$,

Assume $\beta_4 = 0$ and $\beta_3 = 0$.

7 Then $m = 2^{\beta_1} 3^{\beta_2} \in \left\{ 1, 2^2, 2^4, 2^6, \underset{3^2}{\underset{\shortparallel}{9}}, \underset{2^2 \cdot 3^2}{\underset{\shortparallel}{36}}, \underset{3^4}{\underset{\shortparallel}{81}} \right\}$

We get a total of $1 + 2 + 7 = 10$ such integers.

5 (b) How many positive integers x have the property that

$$\operatorname{lcm}(6^6, 8^8, x) = \underbrace{12^{12}}$$

$$6^6 = 2^6 \cdot 3^6 \qquad 8^8 = (2^3)^8 = 2^{24} \qquad (2^2 \cdot 3)^{12} = 2^{24} \cdot 3^{12}$$

$x \mid 12^{12} \Rightarrow$ $x = 2^{e_1} \cdot 3^{e_2}$, where $0 \le e_1 \le 24$, $0 \le e_2 \le 12$,

$\max\{6, 24, e_1\} = 24$,

$\max\{6, 0, e_2\} = 12$,

So, $e_2 = 12$, and $\underbrace{0 \le e_1 \le 24}_{25 \text{ possibilities}}$

So $x = 2^{e_1} \cdot 3^{12}$ and there are 25 such x.