

Name: My Sol'm

1. (35 points) Compute the following (in cartesian or polar form):

a)  $\frac{7}{7}$  pts  
 The polar form of  $z := \frac{3}{1+i}$  =  $\frac{3}{2}(1-i) = \frac{3}{\sqrt{2}} e^{-\frac{\pi i}{4}}$

$\frac{7}{7}$  pts  
 b)  $|z^5|$ , where  $z$  is given in part a.

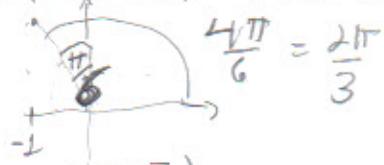
$$\left(\frac{3}{\sqrt{2}}\right)^5$$

$\frac{8}{8}$  pts  
 c)  $\log(z^9)$ , where  $z$  is given in part a.

$$\ln\left(\left(\frac{3}{\sqrt{2}}\right)^9\right) + i \underbrace{\text{Arg}\left(e^{-\frac{9\pi i}{4}}\right)}_{-\frac{\pi}{4}},$$

$$9 \ln\left(\frac{3}{\sqrt{2}}\right)$$

8 pts

d) Find all values of  $(-1 + \sqrt{3}i)^{(1/5)}$ . How many different values are there?

$$2 \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) = 2 e^{i \left( \frac{2\pi}{3} \right)}$$

$\zeta_1 = 2^{1/5} e^{i \left( \frac{2\pi}{15} \right)}$  is one particular root.

$$2^{1/5} e^{i \left( \frac{2\pi}{15} + \frac{2k\pi}{5} \right)}, k=0, 1, 2, 3, 4$$

are all five roots.

8 pts

e) Find all values of  $(1-i)^{(1+i)}$ . How many different values are there?

$$= e^{(1+i) \ln(1-i)} = e^{\ln(\sqrt{2}) + i \left( \frac{\pi}{4} + 2k\pi \right)}$$



$$\ln(\sqrt{2}) + \frac{\pi}{4} + 2k\pi + i \left( \ln(\sqrt{2}) - \frac{\pi}{4} - 2k\pi \right)$$

$$= \left( \sqrt{2} \cdot e^{\frac{\pi}{4}} \cdot e^{i \left[ \ln(\sqrt{2}) - \frac{\pi}{4} \right]} \right) \cdot e^{2k\pi}$$

↑  
 }  $\infty$ -many distinct  
 values for distinct values of  $k$ .

2. (15 points) Prove that the following function is analytic on the whole complex plane  
 $f(x+iy) = \underbrace{(e^{x+y} + e^{-x-y})}_{u} \cos(y-x) + i \underbrace{(e^{x+y} - e^{-x-y})}_{v} \sin(y-x)$ . Carefully state any theorem you use.

Theorem: Let  $f(x+iy) = u(x,y) + i v(x,y)$  be defined on an open set  $U$  over which  $u, v$  have continuous partials and satisfy the Cauchy-Riemann equations  $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ . Then  $f$  is analytic on  $U$ .

In our case:

$$u_x = (e^{x+y} - e^{-x-y}) \cos(y-x) + (e^{x+y} + e^{-x-y}) \sin(y-x)$$

$$v_y = ( \quad ) \quad "$$

$$\text{So } u_x = v_y \\ u_y = (e^{x+y} - e^{-x-y}) \cos(y-x) - (e^{x+y} + e^{-x-y}) \sin(y-x)$$

$$v_x = (e^{x+y} - e^{-x-y})(-\cos(y-x)) + (e^{x+y} + e^{-x-y}) \sin(y-x)$$

$$\text{So } v_x = -u_y,$$

The partials above are continuous on the whole plane. Hence  $f$  is analytic on the whole plane by the above theorem.

3. (a) (5 points) Let  $\lambda$  be a complex number. Show that the equation  $w + \frac{1}{w} = 2\lambda$  in the unknown  $w$  has at least one non-zero complex solution and at most two such solutions. Express the solutions in terms of  $\lambda$ . Hint: reduce to a quadratic equation.

$\textcircled{*}_\lambda$

$$w^2 - 2\lambda w + 1 = 0$$

$$w_{1,2} = \frac{2\lambda \pm \sqrt{4\lambda^2 - 4}}{2} = \lambda \pm \sqrt{\lambda^2 - 1}$$

- (b) (15 points) Show that the function  $\cos : \mathbb{C} \rightarrow \mathbb{C}$  is onto, i.e., that every complex number  $\lambda$  is a value of  $\cos$ . Hint: Use part 3a.

$\textcircled{**}$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \lambda$$

$$e^{iz} + e^{-iz} = 2\lambda$$

$$\begin{matrix} \text{let} \\ w \end{matrix}$$

$z$  satisfies  $\textcircled{**}$  if and only if  $w := e^{iz}$  satisfies  $\textcircled{a}$ .

Let  $w_1$  be a solution of  $\textcircled{a}$ . Then  $w_1 \neq 0$ . The function  $e^z$  maps  $\mathbb{C}$  onto  $\mathbb{C} \setminus \{0\}$  and multiplication by  $i$  maps  $\mathbb{C}$  onto  $\mathbb{C}$ . Hence,  $e^{iz}$  maps  $\mathbb{C}$  onto  $\mathbb{C} \setminus \{0\}$ . Thus, there exists a complex number  $z_1$ , such that  $e^{iz_1} = w_1$ . It follows that  $\cos(z_1) = \lambda$ .

4. (20 points) a) Prove that the function  $u(x, y) = \tilde{u}(x, y) = x^3 - 3xy^2 + 6x^2y - 2y^3 + \ln(x^2 + y^2)$  is harmonic on  $\mathbb{R}^2 \setminus \{(0, 0)\}$  (on the plane minus the origin).

$$u_x = 3x^2 - 3y^2 + 12xy + \frac{2x}{x^2 + y^2} = 2y^2 - 2x^2$$

$$u_{xx} = 6x + 12y + \frac{(2(x^2 + y^2) - 2x(2x))}{(x^2 + y^2)^2} =$$

$$u_y = -6xy + 6x^2 - 6y^2 + \frac{2y}{(x^2 + y^2)}$$

$$u_{yy} = -6x - 12y + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$u_{xx} + u_{yy} = 0,$$

The second partials of  $u$  are rational functions defined and hence continuous in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , so  $u$  is harmonic.

- b) Find a harmonic conjugate  $v$  of the function  $u(x, y)$  in the upper half plane  $\mathbb{H} := \{x + iy \text{ such that } y > 0\}$ . Hint:  $x^2 + y^2 = |z|^2$ .

$\log(z) = \ln(\sqrt{x^2 + y^2}) + i \operatorname{Arg}(z)$  is analytic.

$\log(z) = \ln(x^2 + y^2) + i 2 \operatorname{Arg}(z)$ . So  $2 \operatorname{Arg}(z)$  is a

harmonic conjugate of  $\ln(x^2 + y^2)$ . [Recall that  $V$  is a harmonic conjugate of  $u$  if  $u + iv$  is analytic, by DEFINITION.]

If  $\tilde{v}$  is a harmonic conjugate of  $\tilde{u}(x, y) = x^3 - 3xy^2 + 6x^2y - 2y^3$ , then  $\tilde{v} + 2 \operatorname{Arg}(z)$  is a harmonic conj of  $u = \tilde{u} + \ln(x^2 + y^2)$ .

$$\tilde{v}_y = \tilde{u}_x = 3x^2 - 3y^2 + 12xy$$

$$\tilde{v} = \int (3x^2 - 3y^2 + 12xy) dy + h(x) = 3x^2y - y^3 + 6xy^2 + h(x)$$

$$\tilde{v}_x = 6xy + 6y^2 + h'(x)$$

$$\tilde{v}_x = -\tilde{u}_y = -(-6xy + 6x^2 - 6y^2) = 6xy + 6y^2 - 6x^2$$

$$\text{so } h'(x) = 6x^2 \Rightarrow h(x) = -2x^3 + C$$

$$\text{Take } \tilde{v}(x, y) = 3x^2y - y^3 + 6xy^2 - 2x^3$$

$$v(x, y) = \tilde{v}(x, y) + 2 \operatorname{Arg}(x + iy)$$

In the upper half plane  $\operatorname{Arg}(x + iy) = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$ .

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$$(iy)^3 = -i y^3$$

- c) Find an analytic function  $f(z)$  on the upper half plane  $\mathbb{H}$ , such that  $Re(f)$  is the Harmonic function  $u$  in part a. Your answer must be expressed as a function of  $z = x + iy$ , not  $x$  and  $y$ .

$$f(z) = z^3 - 2iz^3 + 2\log(z)$$

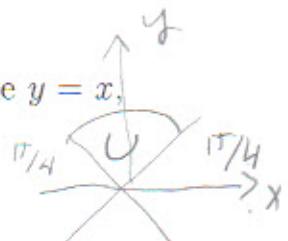
$$\text{Indeed } z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$-2iz^3 = (6x^2y - 2y^3) + i(6xy^2 - 2x^3)$$

$$\text{So } z^3 - 2iz^3 = \tilde{u}(x,y) + i\tilde{v}(x,y)$$

5. (10 points) Let  $U$  be the open subset of the complex plane above the line  $y = x$ , above the line  $y = -x$ , and inside the unit circle

$$U = \{x+iy \text{ such that } y > x \text{ and } y > -x \text{ and } x^2 + y^2 < 1\}.$$

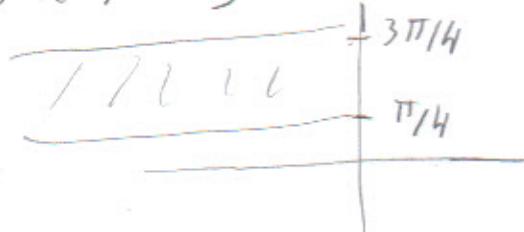


Describe geometrically the image  $\text{Log}(U)$  of  $U$  via the principal branch of the logarithm function. Draw the image and describe it in words. Justify your answer!

$$\text{Log}(x+iy) = \ln(r) + i\theta$$

n.c.  $i\theta$     $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$

Now  $-\infty < \ln(r) < 0$ , since  $r < 1$ . We get



$$\text{Log}(U) = \left\{ x+iy : x < 0, \frac{\pi}{4} < y < \frac{3\pi}{4} \right\}$$