Math 421 Midterm 2 Spring 2012

Name: My solution

Show all your work and justify al your answers!

9 pts

1. (18 points) Let C be the circle of radius 2 centered at the origin and oriented counterclockwise. Evaluate the following integrals.

(a) $\int_C \frac{dz}{\underbrace{z^2+2z-3}_{(z+3)(z-1)}} = \int_C \frac{\left(1/(z+3)\right)}{z-1}\,dz$.

The function $f(z) = \frac{1}{z+3}$ is holo on $\{z : |z|<3\}$ and so on C and at points interior to C.

We get that $\int_C \frac{f(z)}{z-1}\,dz \underset{\boxed{\text{Cauchy's Formula}}}{=} (2\pi i) f(1) = \frac{2\pi i}{4} = \frac{\pi i}{2}$

9 pts

(b) $\int_C Log(z+5)dz$, where $Log(z)$ is the principal branch of the logarithm function with argument in $(-\pi,\pi)$.

The function $Log(z)$ is holo on $\mathbb{C} \setminus \{z : Im(z)=0, Re(z)\leq 0\}$

" " $Log(z+5)$ is holo on $\mathbb{C} \setminus \{z : Im(z)=0, Re(z)\leq -5\}$

Thus, $Log(z+5)$ is holomorphic on C and at all points interior to C. Thus

$$\int_C Log(z+5)\,dz = 0$$

by Cauchy-Goursat's Theorem.

2. (18 points) Let C be the unit circle oriented counterclockwise and let z_0 be a complex number satisfying $|z_0| \neq 1$. Prove the equality

$$\int_C \frac{\sin(z^2)}{(z-z_0)^2} dz = \int_C \frac{2z\cos(z^2)}{z-z_0} dz.$$

(6 pts)

Case 1: If $|z_0| > 1$, then both $\frac{\sin(z^2)}{(z-z_0)^2}$ and $\frac{2z\cos(z^2)}{z-z_0}$ are analytic along C and at all points interior to C. Thus, both sides are zero.

(12 pts)

Case 2: If $|z_0| < 0$, then

$RHS = 2\pi i(2z_0 \cos(z_0^2))$, by Cauchy's Integral Formula.

$LHS = 2\pi i \frac{\partial}{\partial z}\Big|_{z_0} \sin(z^2) = 2\pi i \cos(z_0^2) \cdot 2z_0$,

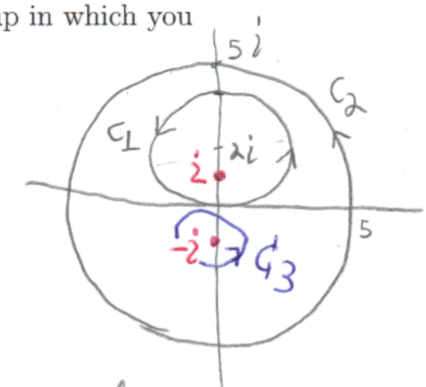
by Cauchy's integral formulas for higher derivatives

3. (10 points) Let $f(z) = e^{(z^2)}\sin(z^4 + z - 2)$. Does f have an anti-derivative? In other words, does there exist an entire function $F(z)$, such that $F'(z) = f(z)$? Carefully justify your answer.

The complex plain $\mathbb{C}$ is simply-connected and $f(z)$ is analytic in the whole of $\mathbb{C}$. Thus, $\int_C f(z)dz = 0$ along any closed contour in $\mathbb{C}$.

By a basic Theorem, f has an anti-derivative.

4. (18 points) Let C_1 be the circle of radius 2 centered at $2i$ oriented counterclockwise. Let C_2 be the circle of radius 5 centered at the origin oriented counterclockwise. Set $f(z) := \frac{1}{(z^2+1)^2}$. Evaluate the difference $\int_{C_2} f(z)dz - \int_{C_1} f(z)dz$.
Hint: Cauchy-Goursat's Theorem for multiply connected regions helps. Clearly state it and explain why its all hypothesis are satisfied in the set-up in which you apply it..



$z^2+1 = (z-i)(z+i)$

Let C_3 be the circle of radius $\frac{1}{2}$ centered at $-i$ and oriented counter clockwise.

The function $f(z)$ is analytic along C_2 and at all points interior to C_2, which are not interior to C_1 nor to C_3. Thus Cauchy-Goursat's theorem for multiply connected domains implies the equality

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_3} f(z)dz,$$

Hence, $\int_{C_2} f(z)dz - \int_{C_1} f(z)dz = \int_{C_3} f(z)dz.$

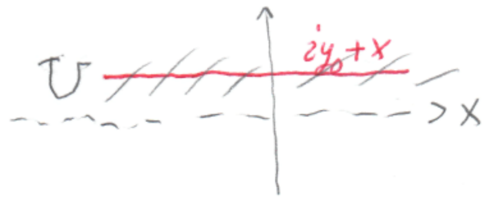
Now,

$$\int_{C_3} f(z)dz = \int_{C_3} \frac{\left[1/(z-i)^2\right]}{(z+i)^2} dz = 2\pi i \left.\frac{\partial}{\partial z}\right|_{z=-i} \frac{1}{(z-i)^2}$$

Cauchy's Integral formula

$$= 2\pi i\,(-2)\frac{1}{(-i-i)^3} = 2\pi i\left(\frac{-2}{8i}\right) = -\frac{\pi}{2}$$

2 pts



5. (18 points) 9 pts

(a) Let U be the upper half-plane $\{x+iy \;:\; y>0\}$ of the complex plane. Set $g(z) := e^{iz}$. Describe geometrically the image $g(U)$ of U under the function g. 3 pts

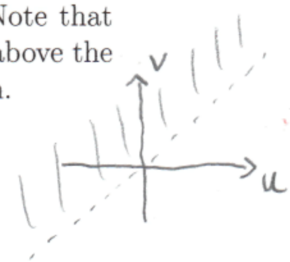
Fix y_0. Then $g(x+iy_0) = e^{i(x+iy_0)} = e^{-y_0+ix} = e^{-y_0}\cdot e^{ix}$
so g maps the horizontal line $y=y_0$ onto the circle of radius e^{-y_0}. Now $y_0>0$, so $0<e^{-y_0}<e^0=1$.
As y_0 varies in $(0,\infty)$, e^{-y_0} varies in $(0,1)$.
We conclude that g maps U onto the punctured unit disk

$$D^* := \{z \;:\; 0<|z|<1\}.$$

Note: e^z maps the left half plane $L := \{x+iy \;:\; x<0\}$ onto D^*.
Multiplication by i rotates U onto L. Hence g maps U onto D^*.

(b) Suppose that $f(z)$ is an entire function. Write $f(x+iy)=u(x,y)+iv(x,y)$. Assume that $v(x,y)\ge u(x,y)$, for all points (x,y) in the plane. Note that the assumption means that the values of f are all in the half-plane above the line $v=u$ in the (u,v) plane. Show that $f(z)$ is a constant function. Hint: Consider the function $g(z)=e^{\lambda f(z)}$, for a suitable constant λ.

9 pts



Denote by V the upper half of the plane consisting of $\{u+iv \;;\; v\geq u\}$.
A point $w=u+iv$ in V has polar form $re^{i\theta}$, $\frac{\pi}{4}<\theta<\frac{5\pi}{4}$.
Set $\lambda = e^{i\pi/4}$. So $\lambda w = re^{i(\theta+\frac{\pi}{4})}$, and

$$\frac{\pi}{2} \leq \theta + \frac{\pi}{4} \leq \frac{6\pi}{4} = \frac{2\pi}{3},$$

So λw belongs to the left half plane $\mathrm{Re}(\lambda w)\leq 0$.

4

So $|e^{\lambda w}| = |e^{\mathrm{Re}(\lambda w)}| \leq |e^0| = 1$,

Now $f(z)$ has values in V, so
$|e^{\lambda f(z)}| \leq 1$, so $e^{\lambda f(z)}$ is a bounded entire function, so $e^{\lambda f(z)}$ is constant, by Liouville's Theorem. Hence f is a constant funct.

4 pts

6. (18 points) Let C_R denote the circle of radius R, $R > 2$, centered at the origin and oriented counterclockwise. Set $I_R := \int_{C_R} \frac{z^2+9}{z^4+3z^2+2}dz$.

9 pts

Let $z = Re^{it}$

(a) Prove the inequality $|I_R| \leq \frac{2\pi R(R^2+9)}{(R^2-1)(R^2-2)}$.

$$|z^2+9| \leq |z^2| + 9 = |z|^2 + 9 = R^2 + 9$$

$$|z^4 + 3z^2 + 2| = |(z^2+1)(z^2+2)| = |z^2+1|\,|z^2+2| \geq$$

$$\geq \big||z^2| - 1\big|\,\big||z^2| - 2\big| = (R^2-1)(R^2-2).$$

Thus, $\left|\frac{z^2+9}{z^4+3z^2+2}\right| \leq \frac{R^2+9}{(R^2-1)(R^2-2)}$

$$I_R \leq \text{length}(C_R) \cdot \frac{R^2+9}{(R^2-1)(R^2-2)} = 2\pi R \frac{(R^2+9)}{(R^2-1)(R^2-2)}$$

4 pts

2 pts

(b) Prove that $\lim_{R\to\infty} I_R = 0$.

$$\lim_{R\to\infty} \frac{2\pi R(R^2+9)}{(R^2-1)(R^2-2)} = \lim_{R\to\infty} \frac{2\pi}{R} \frac{\left(1+\frac{9}{R^2}\right)}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{2}{R^2}\right)} = 0 \cdot \frac{1}{1} = 0,$$

$\frac{2\pi}{R} \to 0$, $\left(1+\frac{9}{R^2}\right) \to 1$, $\left(1-\frac{1}{R^2}\right) \to 1$, $\left(1-\frac{2}{R^2}\right) \to 1$

Now use the sandwitch (squeeze) Theorem and the in equalities $0 \leq |I_R| \leq \frac{2\pi R(R^2+9)}{(\,)(\,)}$,

7 pts

(c) Prove that $I_R = 0$, for all $R \geq 2$.

For all $R \geq 2$, we have $I_R = I_2$, since the function $f(z) = \frac{z^2+9}{(z^2+1)(z^2+2)}$ is analytic on $\mathbb{C} \setminus \{\pm i, \pm\sqrt{2}\, i\}$ and in particular along I_R and at all points interior to C_R and exterior to I_2 (use the principle of deformations of the path.

Now

$0 < |I_2| = |I_R| = \lim_{R\to\infty} |I_R| = 0$. Thus $|I_R| = 0$.

Since constant as a func of R