Math 421 Fall 99 Final

1. Given that the first few terms of the Laurent series for the function $\cot(z)$ around $z = 0$ are:
$$\cot z \;=\; \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots$$
a) Find the principal part at $z = 0$ of the function $f(z) \;=\; \dfrac{(1+z)\cot z}{z^4}$.

   b) Find all the singularities of $f(z)$ in the disk $D = \{|z| < 5\}$. Determine the nature of each singularity (isolated, removable, pole of what order, essential).

   c) Find the residue at each isolated singularity in D.

2. Compute $\displaystyle\int_C \frac{\cos z}{e^{iz}-1} dz$, where C is the circle $\{|z| = 2\}$ (traversed counterclockwise).

3. Compute $I := \int_C (e^{\sin(z)} + \bar{z}) dz$, where C is the circle $\{|z| = 2\}$ (traversed counterclockwise).

4. Compute $I := \displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)}$.

5. Compute $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^6} dx$.

6. **a)** Find the Laurent series of the function $f(z) = \frac{\mathrm{Log} z}{z-i}$ around the point $z_0 = i$.

   b) Find the Taylor series of the function $f(z) = \frac{1}{z^2-3z+2}$ around the point $z_0 = 0$.

7. Determine whether the following statements are true or false. Justify your answers.

   a) The limit $\displaystyle\lim_{z\to 0} \frac{e^{\bar{z}}-1}{z}$ exists and is equal to 1.

   b) There is a function $f(z)$, analytic in the disk $D = \{|z| < 1\}$, such that
$$|f(z)|^2 \;=\; 4 - |z|^2, \quad \text{for all } z \text{ in } D.$$
c) If $f(z)$ has an isolated singularity at z_0 and $\mathrm{Res}_{z=z_0}(f) = 0$, then z_0 is a removeable singularity.

8. Compute $\cos(\frac{\pi}{2} - i\ln 2)$. Simplify your answer as much as possible.

9. Prove that $\left|\displaystyle\int_C e^{iz^2} dz\right| < 5$, where C is the piece of the circle $|z| = 2$ going from 2 to $2i$ counter-clockwise.

10. Find an entire function $f(z)$ such that $\mathrm{Re}(f) \;=\; 4x^3y - 4xy^3 - y$.