1. (36 points) Let $z = \frac{6}{\sqrt{2} - \sqrt{2}i}$. Compute the following (in cartesian or polar form):

a) (8 points) The polar form of $z$ is $z = \frac{6(\sqrt{2} + \sqrt{2}i)}{(\sqrt{2})^2 + (\sqrt{2})^2} = 3e^{\pi i/4}$.

b) (7 points) $|z^3| = |z|^3 = 3^3 = 27$.

c) (7 points) $\log(z^6) = \log(3^6e^{3\pi i/2}) = 6\ln(3) - i\pi/2$.

d) (7 points) The five values of $z^\pi$ are: $3^{1/5}e^{i[\pi/20+2n\pi]/5}$, where $n = 0, 1, 2, 3, 4$.

e) (7 points) The values of $z^i$ are:

$(3e^{i\pi/4})^i = e^{i[\log(3e^{i\pi/4})]} = e^{i[\ln(3)+i(\pi/4+2n\pi i)\pi]} = e^{-\pi/4-2n\pi+i\ln(3)}$, where $n$ is an integer. There are infinitely many such values.

2. (10 points) Let $f(z)$ be an entire function satisfying $|f(z)|^2 = 2$ for all $z$. Prove that $f$ must be a constant function. Hint: Show that the conjugate function $\overline{f(z)}$ must be entire. Then use the Cauchy-Riemann equations to prove that $f'(z) = 0$.

**Method 1:** We first show that $\overline{f(z)}$ is entire (analytic at every point of the plane):
The equality $2 = |f(z)|^2 = f(z) \cdot \overline{f(z)}$ implies, that $\overline{f(z)} = 2/f(z)$ and $f(z)$ does not vanish. By the quotient rule of differentiation, $2/f(z)$ is entire.

Write $f(z) = u(x, y) + iv(x, y)$. Then $\overline{f(z)} = u(x, y) - iv(x, y)$. The Cauchy-Riemann equations (1), (2) for $f(z)$ and (3), (4) for $\overline{f(z)}$ are:

\[
\begin{align*}
  u_x & = v_y \quad \text{and} \\
  u_y & = -v_x, \\
  u_x & = -(v_y) \quad \text{and} \\
  u_y & = -(-v_x).
\end{align*}
\]

Equations (1) and (3) imply that $u_x = u_y = 0$. Equations (2) and (4) imply that $u_y = v_x = 0$. We conclude that $u$ and $v$, and hence also $f$, are constant functions.

**Method 2:** (without showing that $\overline{f(z)}$ is entire). Write $f(z) = u(x, y) + iv(x, y)$. Differentiate both sides of the given equality $u^2 + v^2 = 2$ to get: $2uu_x + 2vv_x = 0$ and $2uv + 2vv_x = 0$. Use the Cauchy-Riemann equations to replace the second equality by an equality involving the partials $u_x$ and $v_x$. We get the system of two linear equations in $u_x$ and $v_x$:

\[
\begin{align*}
  2uu_x + 2vv_x & = 0, \\
  2uv - 2vv_x & = 0,
\end{align*}
\]

whose matrix $\begin{bmatrix} 2u & 2v \\ 2v & -2u \end{bmatrix}$ has determinant $-4(u^2 + v^2) = -8 \neq 0$. Hence, the system has only the trivial solution $u_x = v_x = 0$. The Cauchy-Riemann equations imply also that $u_y = v_y = 0$ and hence $f$ is a constant function.

3. a) (6 points) $\sin(2i) = \frac{e^{i(2i)} - e^{-i(2i)}}{2i} = \frac{e^{-2} - e^2}{2i} = \left[\frac{-e^{-2} + e^2}{2}\right]i$. 

b) (12 points) Find the set of points in the plane, where the function 
\[ f(z) := \frac{z}{\sin(z) - 2i \cos(z)} \] is differentiable. Justify your answer!

**Answer:** The functions \( z, \sin(z), \) and \( \cos(z) \) are entire. Hence, \( \sin(z) - 2i \cos(z) \) is entire, and the quotient \( f(z) \) is analytic at every point, where the denominator does not vanish (the quotient differentiation rule). The points where the denominator vanishes are the solution of \( \sin(z) = 2i \cos(z) \), which, by definition, is

\[ \frac{\cos(z) + i \sin(z)}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = 2i \frac{e^{iz} + e^{-iz}}{2}. \]

Multiply both sides by \( 2i \) and collect the terms to get \( 3e^{iz} = -e^{-iz} \).

Multiply both sides by \( e^{iz} \) to get \( e^{2iz} = -1/3 = 1/3e^{\pi i} \). Set \( z := x + iy \). We get

\[ e^{-2y + 2ix} = \frac{1}{3} e^{\pi i}, \]
or

\[ -2y = \ln(1/3) \quad \text{and} \quad 2ix = i\pi + 2n\pi i. \]

The general solution is \( y = \ln(3)/2 \) and \( x = \frac{\pi}{2} + n\pi \), where \( n \) is an integer.

4. a) (6 points) The function \( u(x, y) = e^x \sin(y) + e^y \cos(x) + 2xy \) is harmonic on the whole of \( \mathbb{R}^2 \), because it satisfies the Laplace equation \( u_{xx} + u_{yy} = 0 \). We verify this by plugging in (or by answering part (c) first).

\[
\begin{align*}
u_x & = e^x \sin(y) - e^y \sin(x) + 2y, \\
u_{xx} & = e^x \sin(y) - e^y \cos(x), \\
u_y & = e^x \cos(y) + e^y \cos(x) + 2x, \\
u_{yy} & = -e^x \sin(y) + e^y \cos(x).
\end{align*}
\]

b) (8 points) The harmonic conjugate \( v \) of the function \( u \) satisfies the two Cauchy-Riemann equations \( v_x = -u_y \) and \( v_y = u_x \). Integrating the second, we get:

\[ v(x, y) = \int u_x \, dy = \int [e^x \sin(y) - e^y \sin(x) + 2y] \, dy = -e^x \cos(y) - e^y \sin(x) + y^2 + h(x). \]

We find \( h'(x) \) using the first equation \( v_x = -u_y \):

\[ -e^x \cos(y) - e^y \cos(x) + h'(x) = -[e^x \cos(y) + e^y \cos(x) + 2x]. \]

Hence, \( h'(x) = -2x, h(x) = -x^2 + C \), and the harmonic conjugate is:

\[ v(x, y) = -e^x \cos(y) - e^y \sin(x) + y^2 - x^2 + C. \]

c) (4 points) Find an entire function \( f(z) \) such that \( \text{Re}(f) = u \).

**Answer:** \( f(z) = -ie^z - e^{-iz} - iz^2 \).

5. a) (9 points) The general point on the horizontal line \( y = 1/4 \) has the form \( z = x + (1/4)i \). The function \( f(z) = e^{\pi z} \) takes this point to \( e^{\pi x + \pi i/4} = e^{\pi x} \cdot \frac{1 + i}{\sqrt{2}} \). The image of the line \( y = 1/4 \) under the function \( f(z) = e^{\pi z} \) is the half-line obtained by multiplying \( 1 + i \) by an arbitrary positive real number, namely the half-line with angle \( \pi/4 \).

b) (9 points) Find the image, under the principal branch of \( \text{Log}(z) \), of the set \( \{ z \text{ such that } |z| < 1 \text{ and } \text{Re}(z) > 0 \} \) (the right half of the unit disk).

**Answer:** A general point in this half-disk has the form \( z = re^{i\theta} \), where \( 0 < r < 1 \), and \( -\pi/2 < \theta < \pi/2 \). Hence \( \text{Log}(z) = \ln(r) + i\theta \), where \( \ln(r) < 0 \). The image is the left-half strip: \( \{ x + iy \text{ such that } x < 0 \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2} \} \).