Math 421 Midterm 2 Fall 2009

Name:________________________

1. (18 points) Compute the integral $\int_C \bar{z}dz$, where C is the triangle with vertices at the points 0, 1, and i, (traversed counterclockwise). *Caution: The integrand is the complex conjugate* $\bar{z}$ *of* z*.*

2. (18 points) Let C be the ellipse cut out by the equation $(x/3)^2 + (y/5)^2 = 1$, oriented counterclockwise. Compute $\int_C \frac{z^3dz}{(z-i)(z^2+1)}$.

3. (16 points) Suppose that $f(z)$ is entire and $|f(z)| \geq 1/2$, for all z in the complex plane. Prove that f is a constant function. Hint: The strategy is similar to the proof of the Fundamental Theorem of Algebra, but the actual proof is much simpler.

4. (16 points) Let C be the unit circle parametrized by $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

   (a) Show that for all integers n, $\int_C (e^{(z^n)}/z)dz = 2\pi i$.

   (b) Derive the integration formula $\int_0^{2\pi} e^{\cos(n\theta)} \cos(\sin(n\theta))d\theta = 2\pi$, for every integer n.

5. (16 points) Let the domain D be the complex plane minus the non-negative part of the x-axis. Let $\log(z)$ be the branch of the logarithm function with argument in the interval $(0, 2\pi)$, so that $log(z)$ is analytic in D. Set $f(z) := e^{(1/2)\log(z)}$. Note that $f(z)$ is a branch of the multi-valued function $\sqrt{z}$.

   (a) Find a single valued anti-derivative $F(z)$ of $f(z)$ in D. Express your answer in terms of the above branch of $\log(z)$ and avoid using multi-valued rational powers of z. Check that your answer is indeed an anti-derivative, by explicitly differentiating it.

   (b) Let C be the contour $z(\theta) = e^{i\theta}$, $\pi/2 \leq \theta \leq 3\pi/2$. Prove the equality

$$\int_C f(z)dz = \frac{2\sqrt{2}}{3}.$$

6. (16 points) Let C be a circle of radius $7/2$ centered at the origin oriented counterclockwise. Set $g(n) := \int_C \frac{z^5+3z+7}{(z-n)^3}dz$. Compute $g(n)$ for all integers n. Justify your answer!!!