# Math 421 Final Exam Fall 2009

Name:________________________

Show **all** your work. Credit will **not** be given for an answer without a justification. Calculators may **not** be used in this exam.

1. (12 points) a) Find all the solutions of $\sin(z) = 0$, for $z$ in the complex plane. Prove your answer!

   b) Is there a positive number $M$, such that the inequality $|\sin(z)| \leq M$ holds for all complex numbers $z$? Justify your answer!

2. (14 points) The Laurent series of $\frac{1}{\sin(z)}$, in some punctured disk centered at 0, has the form

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots \text{terms of order at least seven.} \quad (1)$$

   You are not asked to derive equality (1).

   (a) Find the principal part at $z = 0$ of the function $f(z) = \dfrac{1+z}{z^5 \cdot \sin(z)}$

   (b) Find all the singularities of $f(z)$ (given in part 2a) in the disk $\{z : |z| < 4\}$ and determine their type (isolated, removable, pole of what order, essential). Justify your answer!

   (c) Find the residue of $f$ (given in part 2b) at each isolated singularity in $D$.

3. (10 points) Let $f$ be a continuous function on the closed unit disk $R := \{z : |z| \leq 1\}$, which is analytic in the open unit disk $\{z : |z| < 1\}$. Assume that $f(z) \neq 0$, for all $z$ in $R$. Show that $|f(z)|$ has a minimum $m$ in $R$, which is equal to $|f(z_0)|$, for some $z_0$ with $|z_0| = 1$. Hint: Consider the function $g(z) := 1/f(z)$. Provide a **precise** statement of every theorem you use.

4. (16 points) Compute the following integrals. Show all your work!

   (a) $\displaystyle\int_C \frac{e^z}{z^3 - 2z^2} dz$, where $C$ is the circle $\{z : |z| = 3\}$ traversed counterclockwise.

   (b) $\displaystyle\int_C \frac{\cos(z) + 1}{e^{2z} - e^z} dz$, where $C$ is the circle $\{z : |z| = 1\}$ traversed counterclockwise.

5. (12 points)

   (a) Find the Taylor series of the function $f(z) = \dfrac{2}{z^2 + 4z + 3} = \dfrac{1}{z+1} - \dfrac{1}{z+3}$ centered at $z_0 = 0$. Where is $f(z)$ equal to the sum of its Taylor series? Justify your answer!

   (b) Find the Laurent series representing $f$ in the domain $\{z : 1 < |z| < 3\}$.

6. (10 points) Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin(\theta)}$.

7. (12 points) Let $C_A$ be the straight line segment from $A+iA$ to $-A+iA$, where $A$ is a positive real number, and $A > 1$. Prove the inequality $\left| \int_{C_A} \frac{e^{iz}}{z^2+1} dz \right| \leq \frac{2Ae^{-A}}{A^2-1}$.

8. (12 points) Evaluate the improper integral $\int_0^\infty \frac{dx}{x^4+1}$. Simplify your answer as much as possible. Carefully state any theorem you use.