Problem 34 page 106:

$$(2a+b)^6 = (2a)^6 + \underbrace{\binom{6}{1}}_{6}(2a)^5 b + \underbrace{\binom{6}{2}}_{15}(2a)^4 b^2 + \overbrace{\binom{6}{3}}^{20}(2a)^3 b^3 + \overbrace{\binom{6}{4}}^{15}(2a)^2 b^4 +$$

$$\underbrace{\binom{6}{5}}_{6}(2a) b^5 + b^6$$

$$= (2^6)a^6 + (6\cdot 2^5)a^5 b + (15\cdot 2^4)a^4 b^2 +$$

$$+ (20\cdot 2^3)a^3 b^3 + (15\cdot 2^2)a^2 b^4 + (6\cdot 2)ab^5 + b^6.$$

Problem 45 pag 105:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \underset{\uparrow}{=} (1+1)^n = 2^n$$

Binomial Theorem

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = (1-1)^n = 0^n = 0.$$

Problem 30 page 105:

Find an expression for

$$f(n) \overset{*}{:=} 1 - 3 + 5 - 7 + \cdots + (-1)^{n-1}(2n-1).$$ Prove that it is correct.

Answer:

$f(1) = 1$, $f(2) = -2$, $f(3) = 3$. This suggests that $f(n) \overset{=}{\underset{**}{}} (-1)^{n-1} \cdot n$.

Proof: By induction on n.

$f(1) = 1$ and $(-1)^{1-1} \cdot 1 = 1$. ✓

Induction step: Assume that $f(n) = (-1)^{n-1} \cdot n$.

We need to show ~~that~~ $f(n+1) = \underbrace{(-1)^{(n+1)-1}(n+1)}_{(-1)^n (n+1).}$

$$f(n+1) \overset{\text{By def } *}{=} f(n) + (-1)^{[(n+1)-1]}(2(n+1)-1) \underset{\text{Induction Hyp.}}{=}$$

$$= (-1)^{n-1} \cdot n + (-1)^n (2n+1) = (-1)^n \big[\underbrace{(2n+1) - n}_{n+1}\big] =$$

$$= (-1)^n (n+1).$$

Hence, the equality ** holds for all n, by the Induction Principle. Q.E.D

Problem 57 page 107:

$$D^n(f\cdot g) \overset{(*)}{=} \sum_{k=0}^{n} \binom{n}{k} \left(D^{n-k} f\right)\cdot\left(D^k g\right).$$

<u>Proof:</u> By induction on n.

The case $n=1$ reads

$$\underbrace{D^1(f\cdot g)}_{(fg)'} = \underbrace{\binom{1}{0}}_{1}\underbrace{(D^1 f)}_{f'}\underbrace{(D^0 g)}_{g} + \underbrace{\binom{1}{1}}_{1}\underbrace{(D^0 f)}_{f}\underbrace{(D^1 g)}_{g'} =$$

$$(fg)' = f'g + fg'$$ which holds, by the <u>Product</u> Rule for differentiation.

<u>Induction Step:</u> Note that the definition of the $(n+1)$-st derivative $D^{n+1} f$ is recursive $D^{n+1}(f) = D(D^n f)$. Assume that $(*)$ holds for n.

$$D^{n+1}(f\cdot g) = D\left(D^n(f\cdot g)\right) \underset{\substack{\uparrow \\ \text{IND HYP +} \\ \text{SUM RULE}}}{=} \sum_{k=0}^{n} \binom{n}{k} \underbrace{D\left((D^{n-k} f)\cdot(D^k g)\right)}_{\text{// Product Rule} \\ (D^{n-k+1} f)(D^k g) + (D^{n-k} f)(D^{k+1} g)} =$$

If $0 < k \leq n$ then the term

$(D^{n+1-k} f)(D^k g)$ appears in the above sum twice with coefficients $\binom{n}{k}$ and $\binom{n}{k-1}$. Hence, it appears with coeff $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (Prop 4.32).

The terms $(D^{m+1}f)(D^0 g)$ and $(D^0 f)(D^{m+1}g)$ both appear only once in the above sum with coefficient 1. Hence,

$$D^{m+1}(f \cdot g) = \sum_{k=0}^{m+1} \binom{m+1}{k} (D^{m+1-k}f)(D^k g).$$

Q.E.D