Problem 14 page 104:

$$\underbrace{1^2+3^2+ - + (2m-1)^2}_{\sum_{i=1}^{m}(2i-1)^2} \overset{(*)}{=} \frac{m(2m-1)(2m+1)}{3}$$

Proof:

The base case is $m=1$: $1^2 = \frac{1\cdot\overbrace{(2\cdot1-1)}^{1}\overbrace{(2\cdot1+1)}^{3}}{3}$ ✓

which checks.

Induction step: Assume $(*)$ holds for $m$.

$$\sum_{i=1}^{m+1}(2i-1)^2 = \sum_{i=1}^{m}(2i-1)^2 + (\underbrace{2(m+1)-1}_{2m+1})^2 \underset{\text{Ind. Hyp.}}{=}$$

$$= \frac{m(2m-1)(2m+1)}{3} + (2m+1)^2 =$$

$$= (2m+1)\left[\frac{\overbrace{m(2m-1)+6m+3}^{2m^2+5m+3}}{3}\right] = (2m+1)\frac{(m+1)(2m+3)}{3}$$

$$= \frac{(m+1)(2(m+1)-1)(2(m+1)+1)}{3}$$

The equality thus holds for $m+1$. Hence, the equality holds for all $m$, by the Induction Principle.

Q.E.D

16) $$\underbrace{\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+ \cdots + \frac{n}{2^n}}_{\sum_{i=1}^{n} \frac{i}{2^i}} = 2-\left(\frac{n+2}{2^n}\right) \quad (**)$$

Proof by induction:

Base case ($n=1$): $\frac{1}{2} = 2 - \frac{1+2}{2}$ checks.

Induction Step: Assume that the equality (**) holds for $n$.

$$\sum_{i=1}^{n+1} \frac{i}{2^i} = \left(\sum_{i=1}^{n} \frac{i}{2^i}\right) + \frac{n+1}{2^{n+1}} \underset{\text{Ind. Hyp.}}{=} 2-\left(\frac{n+2}{2^n}\right)+\frac{n+1}{2^{n+1}}$$

$$= 2 - \left[\frac{2n+4}{2^{n+1}} - \frac{n+1}{2^{n+1}}\right] = 2 - \frac{(n+1)+2}{2^{n+1}}.$$

Hence, the equality (**) holds for $n+1$.

QED

Prob 17 page 104:

A set with $n$ elements contains $2^n$ subsets (including the set itself and $\emptyset$).

Proof by induction:

Case $n=1$: If $S$ has 1 element, then its only subsets are $\emptyset$ and $S$ itself, i.e. $2^1$ - subsets.

Induction step: Assume the statement for all sets of $n$ elements. Let $S$ be a set of $(n+1)$ elements and $x \in S$. The subsets of $S$ which do not contain $x$ are the subsets of $S \setminus \{x\}$. The set $S \setminus \{x\}$ has $2^n$ subsets, by the induction hypothesis.

The subsets of $S$ which contain $x$ are all of the form $T \cup \{x\}$, where $T$ is a subset of $S \setminus \{x\}$. Again, by the induction hyp, there are precisely $2^n$ such subsets. Every subset of $S$ either contains $x$ or does not. Hence, $S$ contains $2^n + 2^n = 2^{n+1}$ subsets. We conclude the statement for $S$ as well.

Q.E.D.