61) a) $m^{91} \equiv m^7 \pmod{91}$ for all integers m.

Proof:

Step 1: We first prove the two congruences

(i) $m^{91} \equiv m^7 \pmod{7}$

(ii) $m^{91} \equiv m^7 \pmod{13}$.

Congruence (i) holds if $m \equiv 0 \pmod 7$. Assume $m \not\equiv 0 \pmod 7$.

$m^7 \equiv m \pmod 7$, by Cor 3.44.

Hence,
$$m^{91} \equiv (m^7)^{13} \equiv m^{13} = \underbrace{m^6}_{\equiv 1} \cdot \underbrace{m^6}_{\equiv 1} \cdot m \equiv m.$$
Fermat's Little Thm $\rightarrow$

Hence, congruence (i) holds for all m.

Congruence (ii): $m^{13} \equiv m \pmod{13}$, for all m, by Corollary 3.44. Hence,
$$m^{91} \equiv (m^7)^{13} \equiv m^7 \pmod{13}, \text{ for all } m.$$

Step 2: $m^{91} - m^7 \equiv 0 \pmod 7$ and $\pmod{13}$.

Hence, $m^{91} - m^7 \equiv 0 \pmod{7 \cdot 13 = 91}$.

Hence, $m^{91} \equiv m^7 \pmod{91}$.

b) It is not true that $m^{91} \equiv m \pmod{91}$.

Indeed, if $m^{91} \equiv m \pmod{91}$, then $m^{91} \equiv m \pmod{13}$. But $2^{91} \equiv (2^{13})^7 \equiv \underbrace{2^7}_{128} \equiv -2 \pmod{13}$ and $-2 \not\equiv 2 \pmod{13}$.

$[4]^{-1}$

82) (1) $\underset{\sim}{3}x \equiv 7 \pmod{11}$

(2) $\underset{\underset{-1}{\parallel}}{\underset{\sim}{8}} x \equiv 3 \pmod 9$

$(1) \Leftrightarrow (1)' \quad x \equiv 28 \pmod{11}$

$(2) \Leftrightarrow (2)' \quad x \equiv 6 \pmod 9$,

$(1)'' \quad x = 28 + 11y$

$28 + 11y \equiv 6 \pmod 9$

$\underset{\underset{2}{\parallel}}{\underset{\sim}{11}} y \equiv 5 \pmod 9$

Multiply both sides by $[5] = [2]^{-1}$ to get

$y \equiv 25 \equiv 7 \pmod 9$

Plug 7 for y in $(1)''$ to get the particular sol'n

$x_0 = 28 + 11 \cdot 7 = 105 \ (\equiv 6 \bmod 99)$

The general sol'n is $x \equiv 6 \pmod{99}$.

84) $x^3 \equiv 17 \pmod{99}$

Solve (1) $x^3 \equiv 17 \equiv 8 \pmod 9$

(2) $x^3 \equiv 17 \equiv 6 \pmod{11}$

Mod 9

x	0	1	2	3	4	5	6	7	8
x^3	0	1	$\boxed{8}$	0	1	$\boxed{8}$	0	1	$\boxed{8}$

So, the solutions of (1) are

$x \equiv 2$ or 5 or $8 \pmod 9$

Mod 11

x	0	1	2	3	4	5	6	7	8	9	10
x^3	0	1	8	5	9	4	$-4 \equiv 7$	$-9 \equiv 2$	$-5 \equiv \boxed{6}$	$-8 \equiv 3$	10

So, the only solution of (2) is $x \equiv 8 \pmod{11}$

We need to solve three simultaneous LINEAR congruences:

$$\begin{cases} (1)\ x \equiv 2 \pmod 9 \\ (2)\ x \equiv 8 \pmod{11} \end{cases}$$

$$\begin{cases} (1)'\ x \equiv 5 \pmod 9 \\ (2)'\ x \equiv 8 \pmod{11} \end{cases}$$

$$\begin{cases} (1)''\ x \equiv 8 \pmod 9 \\ (2)''\ x \equiv 8 \pmod{11} \end{cases}$$

The general solution is $\boxed{x \equiv 8 \pmod{99}}$ by Prop 3.6H.

(1) $x \equiv 2 + 9y$

$2 + 9y \equiv 8 \pmod{11}$

$9y \equiv 6 \pmod{11}$

$[9] = [5]^{-1} \pmod{11}$ since $45 \equiv 1 \pmod{11}$

$y \equiv 30 \pmod{11}$

$x_0 \equiv 2 + 9 \cdot 30 = 272 \equiv 74 \pmod{99}$

$$\boxed{x \equiv 74 \pmod{99}}$$

(1)' $x = 5 + 9y$. Plug into (2)' to get

$5 + 9y \equiv 8 \pmod{11}$.

$9y \equiv 3 \pmod{11}$ Multiply both sides by 5

$y \equiv 15 \pmod{11}$

$x_0 = 5 + 9 \cdot 15 = 140 \equiv 41 \pmod{99}$

$$\boxed{x \equiv 41 \pmod{99}}$$

The general solution of $x^3 \equiv 17 \pmod{99}$ is

$$\boxed{x \equiv 8 \text{ or } 74 \text{ or } 41 \pmod{99}.}$$

4)

98) $17^{40} \pmod{\underbrace{27}_{3^3}}$

$\phi(3^3) = 3^2 \cdot 2 = 18$

$\gcd(17, 27) = 1$, so $17^{18} \equiv 1 \pmod{27}$

Hence, $17^{40} \equiv \underbrace{17^4}_{\substack{\equiv \\ (-10)^4 \\ = \\ 10{,}000}} \pmod{27}$

Now $10000 \equiv 10 \pmod{27}$

So $17^{40} \equiv 10 \pmod{27}$.

99) $5^{183} \pmod{99}$

$$\phi(\underbrace{99}_{3^2 \cdot 11}) = \underbrace{\phi(9)}_{6} \cdot \underbrace{\phi(11)}_{10} = 60.$$

$\gcd(5, 99) = 1$. So $5^{\phi(99)} = 5^{60} \equiv 1 \pmod{99}$.

So $5^{183} \equiv (5^{60})^3 \cdot 5^3 \equiv \underbrace{5^3}_{125} \equiv 26 \pmod{99}$.

101) The last two digits of 747^{130}

$$747 = 3^2 \cdot \underset{\underset{-17}{|||}}{83}$$

$\gcd(747, 100) = 1$ and $\phi(100) = 40$

$$\text{So, } 747^{130} = \underbrace{3^{260}}_{\underset{3^{20}}{|||}} \cdot \underbrace{(-17)^{130}}_{\underset{\underset{17^{10}}{|||}}{(17)^{130}}}$$

Binomial Theorem

$$17^{10} = (10+7)^{10} = \underbrace{10^{10} + \binom{10}{1}10^9 \cdot 7 + \cdots + \underbrace{\binom{10}{9}}_{=10}10 \cdot 7^9}_{\underset{0 \text{ mod } (100)}{|||}} + 7^{10}.$$

So $17^{10} \equiv 7^{10} \pmod{100}$

$7^3 = 343 \equiv 43 \pmod{100}$

$7^6 = (43)^2 \equiv 1849 \equiv 49 \pmod{100}$

$7^9 \equiv 7^3 \cdot 7^6 \equiv 43 \cdot 49 = 2107 \equiv 07 \pmod{100}$

$7^{10} \equiv 7^9 \cdot 7 \equiv 49 \pmod{100}$.

$3^2 = 9$

$3^4 = 81$

$3^8 = 6561 \equiv 61 \pmod{100}$

$3^{16} \equiv (61)^2 = 3721 \equiv 21 \pmod{100}$

$3^{20} \equiv 3^4 \cdot 3^{16} \equiv 81 \cdot 21 \equiv 1 \pmod{100}$

So, $747^{130} \equiv 3^{20} \cdot 17^{10} \equiv 3^{20} \cdot 7^{10} \equiv 1 \cdot 49 \equiv 49 \pmod{100}$.