Problem Set 6 page 153:

Exercise 6: $f(x) = \sqrt{x-2}$

The largest set $X \subset \mathbb{R}$ for which $f: X \to \mathbb{R}$ defines a realvalued function is

$$X = \{x \in \mathbb{R} : x \geq 2\}.$$

The image $Y = f(X)$ is

$$\{y \in \mathbb{R} : y \geq 0\}.$$

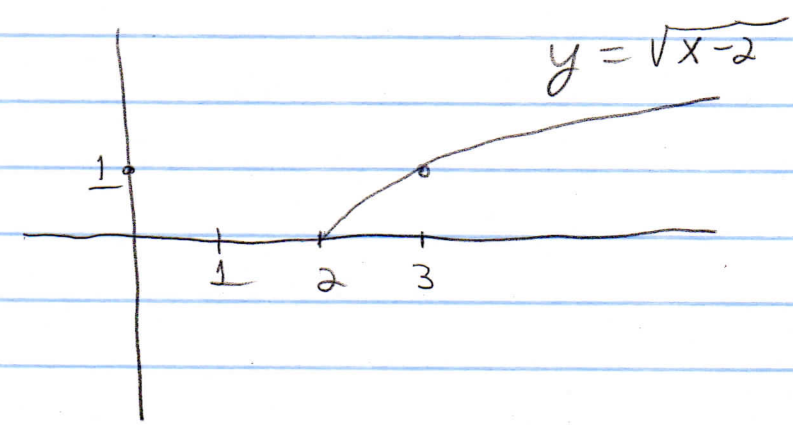
$\underset{[2,\infty)}{f : X} \to \underset{[0,\infty)}{Y}$ is bijective and

$$f^{-1}(y) = y^2 + 2.$$

Check $f(f^{-1}(y)) = f(y^2+2) = \sqrt{(y^2+2)-2} = y.$

$$f^{-1}(f(x)) = f^{-1}(\sqrt{x-2}) = (\sqrt{x-2})^2 + 2 = x.$$

Graph:



Exercise 34:

$X = \mathbb{R} \setminus \{0,1\}$.

Let 1_X be the function $1_X(x) = x$.

$f(x) = 1-x$, $g(x) = \frac{1}{x}$, $g \circ g \overset{(*)}{=} 1_X$, $f \circ f \overset{(**)}{=} 1_X$

$$(f \circ g)(x) = f\left(\frac{1}{x}\right) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

$$(g \circ f)(x) = g(1-x) = \frac{1}{1-x}$$

$$(f \circ g \circ f)(x) = f\left(\frac{1}{1-x}\right) = 1 - \frac{1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$$

$$(g \circ f \circ g)(x) = g\left(\frac{x-1}{x}\right) = \frac{x}{x-1}$$

$$(+) \quad (f \circ g \circ f \circ g)(x) = f\left(\frac{x}{x-1}\right) = 1 - \frac{x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$$
$$= (g \circ f)(x)$$

$$(++) \quad (g \circ f \circ g \circ f)(x) = g\left(\frac{x}{x-1}\right) = \frac{x-1}{x} = (f \circ g)(x).$$

Conclusion: Every composition of the functions f and g is equal to one of the six functions $F = \{1_X, f, g, f \circ g, g \circ f, f \circ g \circ f\}$ whose formulas are given above.

Proof: Let $h = h_1 \circ h_2 \circ \cdots \circ h_n$ be a composition where each h_i is either f or g.

We prove by induction on n, that h is equal to one of the six functions in the set F above.

Initical case: If $n=1$, then $h=h_1=f$ or g and we are done.

Induction step: Assume that

$h=h_1 \circ h_2 \circ \cdots \circ h_K$ belongs to the above set F of six functions, if each h_i is either f or g and $K \leq n$.

Let $h=h_1 \circ h_2 \circ \cdots \circ h_n \circ h_{n+1}$, where each h_i is either f or g. Case 1: If $h_i=h_{i+1}$, for some $1 \leq i \leq n$, then $h_i \circ h_{i+1} = f \circ f = 1_X$ or $h_i \circ h_{i+1} = g \circ g = 1_X$.

Hence, $h=h_1 \circ \cdots \circ h_{i-1} \circ h_{i+2} \circ \cdots \circ h_{n+1}$ and the right hand side is the composition of $n-1$ functions and hence belongs to the set F, by the induction hypothesis.

Case 2: Assume $h_i \neq h_{i+1}$, for $1 \leq i \leq n$.
If $n \leq 3$, then the statement holds, since $f \circ g \circ f = g \circ f \circ g$.
If $n \geq 4$, then $h_1 \circ h_2 \circ h_3 \circ h_4$ is either $f \circ g \circ f \circ g$ or $g \circ f \circ g \circ f$. The former is equal to $g \circ f$ and the latter to $f \circ g$, by (+) and (H).

Hence, we can write h as a composition of $n-1$ functions each equal to f or g, and $h \in F$ by the Ind. Hyp. Q.E.D