Exercise Set 2

1–8. Find the quotient and remainder when b is divided by a in each of the following cases.

1. $a = 3$, $b = 13$

2. $a = 13$, $b = 3$

3. $a = 7$, $b = 7$

4. $a = 7$, $b = 0$

5. $a = 4$, $b = -12$

6. $a = 4$, $b = -10$

7. $a = 11$, $b = -246$

8. $a = 17$, $b = -5$

9. If $3p^2 = q^2$, where $p, q \in \mathbb{Z}$, show that 3 is a common divisor of p and q.

10. If $ac|bc$ and $c \neq 0$, prove that $a|b$.

11. Prove that $\gcd(ad, bd) = |d| \cdot \gcd(a, b)$.

12–18. Find the greatest common divisor of each pair of integers.

12. 5280 and 3600

13. 484 and 451

14. 616 and 427

15. 1137 and -419

16. 19201 and 3587

17. 2^{100} and 100^2

18. $10!$ and 3^{10}

19–26. In each case write $\gcd(a, b)$ in the form $ax + by$, where $x, y \in \mathbb{Z}$.

19. $a = 484$, $b = 451$

20. $a = 5280$, $b = 3600$

21. $a = 17$, $b = 15$

22. $a = 5$, $b = 13$

23. $a = 100$, $b = -35$

24. $a = 3953$, $b = 1829$

25. $a = 51$, $b = 17$

26. $a = 431$, $b = 0$

27. Prove that $\gcd(a, c) = \gcd(b, c) = 1$ if and only if $\gcd(ab, c) = 1$.

28. Prove that any two consecutive integers are relatively prime.

29. Simplify

$$\frac{95}{646} + \frac{40}{391}.$$

30. Gear A turns at 1 rev/min and is meshed into gear B. If A has 32 teeth and B has 120 teeth, how often will both gears be simultaneously back in their starting positions?

31–36. Find one integer solution, if possible, to each Diophantine equation.

31. $21x + 35y = 7$

32. $14x + 18y = 5$

33. $x + 14y = 9$

34. $11x + 15y = 31$

35. $143x + 253y = 156$

36. $91x + 126y = 203$

37–42. Find all the integer solutions to each Diophantine equation.

37. $7x + 9y = 1$

38. $212x + 37y = 1$

39. $15x - 24y = 9$

40. $16x + 44y = 20$

41. $243x + 405y = 123$

42. $169x - 65y = 91$

43–46. Find all the ***nonnegative*** *integer solutions to each Diophantine equation.*

43. $14x + 9y = 1000$

44. $12x + 57y = 423$

45. $38x + 34y = 200$

46. $11x - 12y = 13$

47. Can 1000 be expressed as the sum of two positive integers, one of which is divisible by 11 and the other by 17?

48. Can 120 be expressed as the sum of two positive integers, one of which is divisible by 11 and the other by 17?

49. Can 120 be expressed as the sum of two positive integers, one of which is divisible by 14 and the other by 18?

50. Find the smallest positive integer x so that $157x$ leaves remainder 10 when divided by 24.

51. The nickel slot of a pay phone will not accept coins. Can a call costing 95 cents be paid for exactly using only dimes and quarters? If so, in how many ways can it be done?

52–54. Convert the following numbers to base 10.

52. $(5613)_7$

53. $(100110111)_2$

54. $(9A411)_{12}$, where A is the symbol for ten.

55. How many seconds are there in 4 hours 27 minutes and 13 seconds?

56–61. Convert the following numbers to the indicated base.

56. 1157 to base 2

57. 1241 to base 9

58. 433 to base 5

59. 30 to base 3

60. 5766 to base 12, writing A for ten and B for eleven

61. 40239 to base 60

62. Add and multiply $(1011)_2$ and $(110110)_2$ together in base 2.

63. Add and multiply $(3130)_4$ and $(103)_4$ together in base 4.

64. Write out the addition and multiplication tables for base 6 arithmetic, and then multiply $(4512)_6$ by $(343)_6$ in base 6.

65. Subtract $(3321)_4$ from $(10020)_4$ in base 4, and check your answer by converting to base 10.

66. If $a = (342)_8$ and $b = (173)_8$, find $a - b$ without converting to base 10. [If you get stuck, listen to the song "The New Math" by Tom Lehrer on the album *That Was the Year That Was*.]

67. How many positive divisors does 12 have?

68. How many positive divisors does 6696 have?

69. If we wish to add the fractions $\frac{1}{132} + \frac{4}{9}$, what is the smallest common denominator we could choose?

70–71. *Factor the following numbers into prime factors and calculate the greatest common divisor and least common multiple of each pair.*

70. 40 and 144

71. 5280 and 57800

72. Find $\text{lcm}(12827, 20099)$.

Problem Set 2

73. Prove that $\{ax + by \mid x, y \in \mathbb{Z}\} = \{n \cdot \gcd(a, b) \mid n \in \mathbb{Z}\}$.

74. Show that $\gcd(ab, c) = \gcd(b, c)$ if $\gcd(a, c) = 1$. Is it true in general that

$$\gcd(ab, c) = \gcd(a, c) \cdot \gcd(b, c)\,?$$

75. Show that the Diophantine equation $ax^2 + by^2 = c$ does not have any integer solutions unless $\gcd(a, b)|c$. If $\gcd(a, b)|c$, does the equation always have an integer solution?

76. For what values of a and b does the Diophantine equation $ax + by = c$ have an infinite number of positive solutions for x and y?

77. For what values of c does $8x + 5y = c$ have exactly one strictly positive solution?

78. An oil company has a contract to deliver 100000 liters of gasoline. Their tankers can carry 2400 liters and they can attach one trailer carrying 2200 liters to each tanker. All the tankers and trailers must be completely full on this contract, otherwise the gas would slosh around too much when going over some rough roads. Find the least number of tankers required to fulfill the contract. Each trailer, if used, must be pulled by a full tanker.

79. A trucking company has to move 844 refrigerators. It has two types of trucks it can use; one carries 28 refrigerators and the other 34 refrigerators. If it only sends out full trucks and all the trucks return empty, list the possible ways of moving all the refrigerators.

80. Show how to measure exactly 2 liters of water from a river using a 27 liter jug and a 16 liter jug. If you could not lift the larger jug when full but could push it over, could you still measure the 2 liters?

81. Let S be the complete solution set of the Diophantine equation $ax + by = d$. Is

$$cS = \{(cx, cy) \mid (x, y) \in S\}$$

the complete solution set of $ax + by = cd$?

82. Four men and a monkey spend the day gathering coconuts on a tropical island. After they have all gone to sleep at night, one of the men awakens and, not trusting the others, decides to take his share. He divides the coconuts into four equal piles, except for one remaining coconut, which he gives to the monkey. He then hides his share, puts the other piles together, and goes back to sleep. Each of the other men awakens during the night and does likewise, and every time there is one coconut left over for the monkey. In the morning all the men awake, divide what's left of the coconuts into four, and again there is one left over that is given to the monkey. Find the minimum number of coconuts that could have been in the original pile.

83. Let a, b, c be nonzero integers. Their *greatest common divisor* $\gcd(a, b, c)$ is the largest positive integer that divides all of them. Prove that

$$\gcd(a, b, c) = \gcd(a, \gcd(b, c)).$$

84. Prove that the Diophantine equation $ax + by + cz = e$ has a solution if and only if $\gcd(a, b, c) | e$.

85. If $\gcd(a, b, c) | e$, describe how to find one solution to the Diophantine equation $ax + by + cz = e$.

86. Describe how to find all the solutions to the Diophantine equation

$$ax + by + cz = e.$$

87. Find one integer solution to the Diophantine equation $18x + 14y + 63z = 5$.

88. Find all the ways that \$1.67 worth of stamps can be put on a parcel, using 6 cent, 10 cent, and 15 cent stamps.

89. Given a balance and weights of 1, 2, 3, 5, and 10 grams, show that any integer gram weight up to 21 grams can be weighed. If the weights were 1, 2, 4, 8, and 16 grams, show that any integer weight up to 31 grams could be weighed.

90. If weights could be put on either side of a balance, show that any integer weight up to 121 grams could be weighed using weights of 1, 3, 9, 27, and 81 grams.

91. If numbers (in their decimal form) are written out in words, such as six hundreds, four tens, and three for 643, we require one word for each digit 0, 1, 2, ..., 9, one word for 10, and one word for 10^2, and so on. We can name all the integers below 1000 with twelve words. What base would use the least number of words to name all the numbers below 1000? What base would use the least number of words to name all the numbers below 10^6?

92. Consider the set of all even integers $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$. We can add, subtract, and multiply elements of $2\mathbb{Z}$, and the result will always be in $2\mathbb{Z}$, but we cannot always divide. We can define divisibility and factorization in $2\mathbb{Z}$ in a similar way to that in $\mathbb{Z}$. (For example, $2|4$ in $2\mathbb{Z}$, but $2 \nmid 6$ even though $6 = 2 \cdot 3$, because $3 \notin 2\mathbb{Z}$.) A prime in $2\mathbb{Z}$ is a positive even integer that cannot be factored into the product of two even integers.

(a) Find all the primes in $2\mathbb{Z}$.
(b) Can every positive element of $2\mathbb{Z}$ be expressed as a product of these primes?
(c) If this factorization into primes can be accomplished, is it unique?

93. Prove that the sum of two consecutive odd primes has at least three prime divisors (not necessarily different).

94. How many zeros are there at the right end of

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1\,?$$

95. Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

can never be an integer if $n > 1$.

96. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x (that is, the integer part of x), then for which values of n does $\lfloor \sqrt{n} \rfloor$ divide n?

97. Let a and b be integers greater than 1, and let $e = \operatorname{lcm}(a, b)$. Prove that

$$0 < \frac{1}{a} + \frac{1}{b} - \frac{1}{e} < 1.$$

98. If a and b are odd positive integers, and the sum of the integers, less than a and greater than b, is 1000, then find a and b.

99–102. *Either prove each of the following statements about integers or give a counterexample.*

99. $a^2|b^2$ if and only if $a|b$
100. $\gcd(a, b) = \gcd(a + b, \operatorname{lcm}(a, b))$
101. $\operatorname{lcm}(\gcd(a, b), \gcd(a, c)) = \gcd(a, \operatorname{lcm}(b, c))$
102. If $\gcd(a, b) = 1$ and $ax + by = c$ has a positive integer solution, then so does $ax + by = d$ when $d > c$.

103. Write a computer program to test whether a given number is prime. Use your program to find the smallest positive integer n for which the number $n^2 - n + 41$ fails to be prime.

104. Using a computer, test whether $F(4) = 2^{2^4} + 1$ and $F(5) = 2^{2^5} + 1$ are prime.

105. Show that all the integers, $\mathbb{Z}$, both positive and negative, can be represented in the *negative base* -10 using the digits $0, 1, \ldots, 9$ without using a negative prefix. For example, $-1467 = (2673)_{-10}$ and $10 = (190)_{-10}$.

(a) What decimal numbers do $(56)_{-10}$ and $(164)_{-10}$ represent?

(b) Find the negative ten representations of the decimal numbers 1111 and -209.

(c) Try adding and multiplying some numbers in the base negative ten. Then try adding a number to its negative.

106. **(a)** Find two consecutive primes that differ by at least 10.

(b) Prove that there are arbitrarily large gaps between consecutive primes.

107. Let $a < b < c$, where a is a positive integer and b and c are odd primes. Prove that if $a \mid (3b + 2c)$ and $a \mid (2b + 3c)$, then $a = 1$ or 5. Give examples to show that both these values for a are possible.

108. An integer n is *perfect* if the sum of its divisors (including 1 and itself) is $2n$. Show that if $2^p - 1$ is a prime number, then $n = 2^{p-1}(2^p - 1)$ is perfect.