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Math 300 Section 2 Midterm 1 Spring 2016

Name:__________________________

1. (15 points) Let P and Q be statements. Find the truth tables of the following statements and use them to determine if they are equivalent.
   i) $P \Rightarrow Q$.
   ii) (NOT Q) $\Rightarrow$ (NOT P).

2. (10 points) Let the universe of discourse be the real numbers. Prove or give a counter example to the following statement.
   $\forall x \ \exists y \ (x^2 > y^2)$.

3. (10 points) Let the universe of discourse be the real numbers. Write the contrapositive of the following statement.
   If $x < -2$, then $x^2 > 4$.

4. (15 points) Let S and T be sets. Use the contrapositive method to prove the following statement.
   $((S \cap T = \emptyset)$ AND $(S \cup T = T)) \Rightarrow (S = \emptyset)$.
   Suggestion: State the equivalent contrapositive statement and start your argument with: "Assume that $S \neq \emptyset$. Then there exists an element $x \in S$. ..."

5. (10 points) How many positive divisors does 360^3 has? Justify your answer!

6. (10 points) Let x, y, z, and w be positive integers. Assume that $\gcd(x, z) = 1$ and that $y|z$. Prove that if $x|yw$, then $x|w$.

7. (15 points) a) Use the Extended Euclidean Algorithm to find a particular solution of the equation $33x + 18y = \gcd(33, 18)$.

   b) Find all the integer solutions of the equation $33x + 18y = 150$. Show all your work.

   c) Find all positive integer solutions of the equation $33x + 18y = 150$.

8. (15 points) a) Let x be a positive integer and $x = p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}$ its prime decomposition. What can you say about the parity (even or odd) of the exponents in the prime decomposition of x^2?

   b) Let p be a prime. Prove, by contradiction, that the equation $px^2 = y^2$ does not have any positive integer solutions x, y. Hint: Use part **??**a.

   c) (Bonus 5 points) Prove, by contradiction, that if p is prime then $\sqrt{p}$ is not a rational number.

1. (15 points) Let P and Q be statements. Find the truth tables of the following statements and use them to determine if they are equivalent.
   i) $P \Rightarrow Q$.
   ii) (NOT Q) $\Rightarrow$ (NOT P).

P	Q	$P \Rightarrow Q$	Not P	Not Q	Not $Q \Rightarrow$ Not P
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

The statements (i) and (ii) are equivalent,

P	Q	R	- - - - -
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

2. (10 points) Let the universe of discourse be the real numbers. Prove or give a counter example to the following statement.
$\forall x \; \exists y \; (x^2 > y^2)$.

$\text{Not}\left(\forall x \; \exists y \; (x^2 > y^2) \right)$

$\iff$

(*) $\exists x \; \forall y \; x^2 \leq y^2$.

Indeed $x = 0$ is an example.

So (*) is true,

So the original statement is False,

3. (10 points) Let the universe of discourse be the real numbers. Write the contrapositive of the following statement.
If $\underbrace{x < -2}_{P}$, then $\underbrace{x^2 > 4}_{Q}$.

If $\underbrace{x^2 \leq 4}_{\text{Not } Q}$, then $\underbrace{x \geq -2}_{\text{Not } P}$.

4. (15 points) Let S and T be sets. Use the contrapositive method to prove the following statement.
$((S \cap T = \emptyset) \text{ AND } \underbrace{(S \cup T = T)}_{Q}) \Rightarrow \underbrace{(S = \emptyset)}_{R}$.
Suggestion: State the equivalent contrapositive statement and start your argument with: "Assume that $S \neq \emptyset$. Then there exists an element $x \in S$. ..."

P

We will prove the equivalent contrapositive statement:

"If $\underbrace{S \neq \emptyset}_{\text{Not } R} \Rightarrow ((S \cap T \neq \emptyset) \text{ OR } (S \cup T \neq T))$".

Not P OR Not Q

Assume that $S \neq \emptyset$. Let x be an element in S. If $x \in T$, then x belongs to both S and T, so $x \in S \cap T$, so $S \cap T \neq \emptyset$.

If $x \notin T$, then $(x \in S \cup T$ and $x \notin T)$ so $S \cup T \neq T$.

Q.E.D

5. (10 points) How many positive divisors does 360^3 has? Justify your answer!

Step 1: Find the prime factorization of 360^3.

$$360 = \underbrace{10}_{2\cdot 5} \cdot \underbrace{36}_{\underbrace{6^2}_{2^2\cdot 3^2}} = 2^3\, 3^2\, 5^1$$

$$360^3 = \left(2^3 \cdot 3^2 \cdot 5^1\right)^3 = 2^9 \cdot 3^6 \cdot 5^3.$$

Step 2: Every divisor d of 360^3 is of the form

$$d = 2^{d_1} \cdot 3^{d_2} \cdot 5^{d_3},$$

where

$0 \le d_1 \le 9$, 10 possibilities

$0 \le d_2 \le 6$ 7 "

$0 \le d_3 \le 3$ 4 "

360^3 has $\underbrace{10 \cdot 7 \cdot 4}_{280}$ divisors

6. (10 points) Let x, y, z, and w be positive integers. Assume that $\gcd(x, z) = 1$ and that $y|z$. Prove that if $x|yw$, then $x|w$.

Recall the following Proposition from class (and textbook),

Proposition: Let x, y, w be integers. If $\gcd(x, y) = 1$ and $x | yw$, then $x | w$.

Assume that $\gcd(x, z) = 1$ and $y | z$, We will prove that $\gcd(x, y) = 1$. Then we conclude that $x | w$, using the above proposition,

Let c be a common divisor of x and y. Then $c | x$ and $c | y$ and we are given that $y | z$. So $c | z$. So c is a common divisor of x, z. So $c \leq 1$ (because $\gcd(x, z) = 1$ by assumption), So $\gcd(x, y) = 1$.

Q.E.D

7. (15 points) a) Use the Extended Euclidean Algorithm to find a particular solution of the equation $33x + 18y = \gcd(33, 18)$.

b) Find all the integer solutions of the equation $33x + 18y = 150$. Show all your work.

c) Find all positive integer solutions of the equation $33x + 18y = 150$.

$50 \cdot \gcd(33,18)$

a) $33x_i + 18y_i = r_i$

x_i	y_i	r_i	q_i
1	0	33	—
0	1	18	—
1	-1	15	$\boxed{1}$
-1	2	$\boxed{3}$	$\boxed{1}$
		0	

$33 = 18 \cdot \overset{q_3}{\boxed{1}} + \overset{r_3}{\boxed{15}}$

$\gcd(33,18) = 3$

$18 = 15 \cdot \overset{q_4}{\boxed{1}} + \overset{r_4}{\boxed{3}}$

$33(-1) + 18 \cdot (2) = 3 = \gcd(33, 18).$

b) One solution of

$(+)$ $33x + 18y = 150 = 50 \cdot 3$

is $(x_0, y_0) = 50(-1, 2) = (-50, 100).$

The general solution of the homogeneous equation

$33x + 18y = 0$ is

$(x_h, y_h) = \{(\frac{18}{3}k, -\frac{33}{3}k)$; k integer

$(x_h, y_h) \in \{(6K, -11K) : K \text{ integer}\}$

The general solution of $(+)$ is

$$(x,y) = (x_0 + x_h, y_0 + y_h) = (-50+6K, 100-11K),$$

where $K \in \mathbb{Z}$.

c) Find all positive solutions of $(+)$.

The integer K should satisfy

$$-50+6K > 0 \quad \text{AND} \quad 100-11K > 0$$

$\Longleftrightarrow$

$6K > 50$

$K > \frac{50}{6} = 8\frac{1}{3}$

$\boxed{K \geq 9}$

$\Longleftrightarrow$

$100 > 11K$

$9\frac{1}{11} = \frac{100}{11} > K$

$\boxed{9 \geq K}$

The only solution is

$$(x,y) = (\underbrace{-50 + \underbrace{6\cdot 9}_{54}}_{4}, \underbrace{100 - \underbrace{11\cdot 9}_{99}}_{1})$$

$(x,y) = (4,1)$.

Check: $33\cdot 4 + 18\cdot 1 \overset{?}{=} 150$ ✓

8. (15 points) a) Let x be a positive integer and $x = p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}$ its prime decomposition. What can you say about the parity (even or odd) of the exponents in the prime decomposition of x^2?

b) Let p be a prime. Prove, by contradiction, that the equation $px^2 = y^2$ does not have any positive integer solutions x, y. Hint: Use part 8a.

c) (Bonus 5 points) Prove, by contradiction, that if p is prime then $\sqrt{p}$ is not a rational number.

a) $x^2 = (p_1^{d_1} \cdots p_n^{d_n})^2 = p_1^{2d_1} \cdots p_n^{2d_n}$

The exponents of p_i are even, for all i,

b) The equation $px^2 = y^2$ does NOT have any positive integer sol'ns x, y (given that p is a prime).

Assume that x, y are integers and $px^2 = y^2$.

The prime p appears with even exponent (possibly zero) in the prime factorization of x^2, by part a,

* So p appears with odd exponent in the prime factorization of px^2

The prime p appears with <u>even</u> exponent in the prime factorization of y^2, by part a. Now $y^2 = px^2$,

$\boxed{\text{so } p \text{ appears with } \underline{\text{even}} \text{ exponent in the prime factorization of } px^2}$ (**)

The prime factorization is unique, so (*) and (**) are contradicting statements.
Hence an integers solution x, y does not exist.

c) Suppose that $\sqrt{p}$ is a rational number. I.e., there exist integers x, y, $x \neq 0$ such that

$$\sqrt{p} = \frac{y}{x}$$

$$\Longleftrightarrow \sqrt{p} \cdot x = y.$$

So squaring both sides, we get

$$(*) \quad p \cdot x^2 = y^2.$$

So (x, y) is an integer solution to the equation. This contradicts part (b).

Q.E.D.