Exercise Set 4

1–4. Calculate the following.

1. $\binom{5}{3}$

2. $\binom{10}{6}$

3. $\dfrac{8!}{(4!)^2}$

4. $100! - 99!$

5. Show that $\dfrac{1}{n}\binom{n}{r} = \dfrac{1}{r}\binom{n-1}{r-1}$.

6. Show that $\binom{n}{r}\binom{r}{s} = \binom{n}{s}\binom{n-s}{r-s}$.

7. Find n if $\binom{n+2}{n} = 36$.

8–10. Write the following in sigma notation.

8. $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{5}{6} + \cdots + \dfrac{99}{100}$

9. $8 + 15 + 24 + 35 + \cdots + (n^2 - 1)$

10. $a^k + a^{2k} + a^{4k} + a^{8k} + a^{16k} + \cdots + a^{256k}$

11–18. Prove, by induction, the following results for all $n \in \mathbb{P}$.

11. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

12. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

13. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \dfrac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$

14. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

15. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

16. $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \left(\dfrac{n+2}{2^n}\right)$

17. A set with n elements contains 2^n subsets (including the set itself and $\emptyset$).

18. $6 | (2n^3 + 3n^2 + n)$

19. Find an expression for $\sum_{r=1}^{n} r(r!)$ and prove that it is correct.

20–23. *Are the following true for all positive integer values of n? If so, prove the result; if not, give a counterexample.*

20. $n! \geq 2n$

21. $3|(2^{2n} - 1)$

22. $7|(5^n + n + 1)$

23. $(a+b)|(a^{2n} - b^{2n})$

24. Prove that the sum of the first n terms of the *arithmetic progression*

$$a + (a+d) + (a+2d) + \cdots + [a + (n-1)d]$$

is $\frac{n}{2}[2a + (n-1)d]$; that is, $\frac{n}{2}$ times the sum of the first and last terms.

25. Prove that the sum of the first n terms of the *geometric progression*

$$a + aq + aq^2 + \cdots + aq^{n-1}$$

is $\dfrac{a(1-q^n)}{1-q}$ when $q \neq 1$.

26. *(Fermat's Little Theorem)* If p is a prime, use induction on n and the Binomial Theorem to prove that $n^p \equiv n \pmod{p}$ for all $n \in \mathbb{P}$.

27. Use induction to prove that $a^m \cdot a^n = a^{m+n}$ for all $n \in \mathbb{P}$.

28. A sequence of integers $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 3$, $x_2 = 7$, and

$$x_k = 5x_{k-1} - 6x_{k-2} \quad \text{for } k \geq 3.$$

Prove that $x_n = 2^n + 3^{n-1}$ for all $n \in \mathbb{P}$.

29. If n points lie in a plane and no three are collinear, prove that there are $\frac{1}{2}n(n-1)$ lines joining these points.

30. Find an expression for

$$1 - 3 + 5 - 7 + 9 - 11 + \cdots + (-1)^{n-1}(2n-1)$$

and prove that it is correct.

31. The notation $\prod_{r=1}^{n} a_r$, which uses the capital Greek letter pi, stands for the *product* of all the terms obtained by substituting the integers from 1 to n for r in the expression a_r. That is,

$$\prod_{r=1}^{n} a_r = a_1 \cdot a_2 \cdot a_3 \cdots a_n.$$

Prove that $\displaystyle\prod_{r=1}^{n}(1 + x^{2^r}) = \frac{1 - x^{2^{n+1}}}{1 - x^2}$, if $x^2 \neq 1$.

$1 - \frac{(M-1)}{2M}$

32. Find an expression for $\displaystyle\prod_{r=2}^{n}\left(1 - \frac{1}{r^2}\right)$ and prove that your expression is correct.

33. Use induction to prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$

(This can be used to show that the infinite harmonic series $\sum_{r=1}^{\infty} \frac{1}{r}$ diverges.)

34–38. Expand the following by the Binomial Theorem.

34. $(2a+b)^6$

35. $(a-1)^5$

36. $\left(x+\dfrac{1}{x}\right)^8$

37. $\left(4x^2-3y^3\right)^4$

38. $(a+b+c)^3$

39. Calculate $(2.99)^4$ to 3 decimal places.

40. Calculate $(1.02)^{10}$ to 3 decimal places.

41. Find the fifth term in the expansion of $\left(2x^6-\dfrac{5}{x^5}\right)^{11}$.

42. Find the term containing x^5 in the expansion of $\left(x^2+\dfrac{3}{x}\right)^6$.

43. If p is a prime, prove that

$$(a+b)^p \equiv a^p + b^p \pmod{p} \qquad \text{for all } a, b \in \mathbb{Z}.$$

44. A man earns a starting salary of \$10,000 a year and receives an annual wage increase of 10%. Find his salary, to the nearest dollar, after 10 years.

45. Prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n$$

and that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n} = 0.$$

Problem Set 4

46. Prove that

$$\sum_{r=1}^{n} rx^r = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

for all $n \in \mathbb{P}$, where x is a number different from 1.

47. Prove that $\sqrt[n]{2} \le 1 + \dfrac{1}{n}$ for all $n \in \mathbb{P}$.

48. Find an expression for $1^2-3^2+5^2-7^2+\cdots+(-1)^{n-1}(2n-1)^2$ and prove that your expression is correct.

49. Prove that a convex n-gon contains $\frac{1}{2}n(n-3)$ diagonals. (A convex n-gon is a polygon with n sides such that all the line segments, joining two nonadjacent vertices, lie inside the polygon.)

50. Generalize the following sequence of identities and prove your generalization.

$$1 = 1, \quad 3 + 5 = 8, \quad 7 + 9 + 11 = 27, \quad 13 + 15 + 17 + 19 = 64$$

51. Prove that the product of r consecutive positive integers is divisible by $r!$.

52. Find an expression for

$$\sum_{i=1}^{n} i(i+1)(i+2)\cdots(i+r-1)$$

and prove by induction that your expression is correct.

53. If $x \equiv 1 \pmod 2$, prove that $x^{2^n} \equiv 1 \pmod{2^{n+2}}$ for all $n \in \mathbb{P}$.

54. Is $2^n - 1$ the minimum number of moves required to solve the Tower of Hanoi puzzle in Example 4.17? Give reasons for your answer.

55. If the three pegs in the Tower of Hanoi puzzle are labeled A, B, and C, and n disks are initially on peg A, then what is the first move in order to transfer the disks to peg B in $2^n - 1$ moves?

56. *(Pigeonhole Principle)* Prove the following statement by induction on n. "If $n + 1$ objects are placed in n boxes (or pigeonholes), then one box must contain at least two objects."

57. *(Leibniz Rule)* If you have some knowledge of the calculus, prove by induction that

$$D^n(f \cdot g) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r} f \cdot D^r g$$

where Df is the derivative of the function f. Assume that all the necessary derivatives exist.

58. What is wrong with the following proof that all horses have the same color? We shall prove the result by induction on the number of horses. Clearly, in any set consisting of a single horse, all horses have the same color. As induction hypothesis, assume that in any set of h horses, they all have the same color. In any set of $h+1$ horses, where $h \geq 1$, remove one horse. By the induction hypothesis, the remaining h horses have the same color. Now put the horse back, and remove a different horse. All the h horses left have the same color. Hence all the $h+1$ horses have the same color. Therefore, by the Principle of Mathematical Induction, all horses have the same color.

59–67. *In 1202 Leonardo of Pisa, also called Fibonacci, published an influential mathematical book that popularized the Indian-Arabic number system in Europe. He used our current Arabic numerals, instead of Roman numerals, and also a positional decimal system with a sign for zero. One of the problems in the book asked "How many pairs of rabbits can be produced in a year from a single pairs of baby rabbits, if every month each mature pair produces a new pair of babies, that mature after a month?" The answer involves the sequence 1, 1, 2, 3, 5, 8, 13, ..., in which each term is the sum of the previous two. This sequence is known as the* ***Fibonacci sequence*** *and is formally defined as the sequence of integers* $f_1, f_2, f_3, \ldots$ *such that* $f_1 = 1$, $f_2 = 1$, *and* $f_n = f_{n-1} + f_{n-2}$, *for* $n \geq 3$.

59. Find the first 15 numbers in the Fibonacci sequence.

60. Prove that $\gcd(f_n, f_{n+1}) = 1$ for all $n \in \mathbb{P}$.

61. Look at the Euclidean Algorithm applied to f_{14} and f_{15}. What do you notice about the sequence of remainders?

62. Prove that $f_{n+1} < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{P}$.

63. If $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$, prove that $f_n = \dfrac{a^n - b^n}{\sqrt{5}}$ for all $n \in \mathbb{P}$.

64. If $a = \dfrac{1+\sqrt{5}}{2}$, prove that f_n is the nearest integer to $\dfrac{a^n}{\sqrt{5}}$ for all $n \in \mathbb{P}$. Hence compute f_{28}, f_{29}, and f_{30}, and verify that $f_{30} = f_{28} + f_{29}$.

65. Prove that $\sum_{r=1}^{n} f_r^2 = f_n f_{n+1}$ for all $n \in \mathbb{P}$.

66. Prove that $\sum_{r=1}^{n} f_{2r-1} = f_{2n}$ for all $n \in \mathbb{P}$.

67. Prove that $f_{n+5} \equiv 3f_n \pmod{5}$ for all $n \in \mathbb{P}$.

68. The downtown portion of a city consists of a rectangular area m blocks long and n blocks wide. If all the streets in each direction are through streets, find the number of different shortest routes from one corner of the downtown area to the opposite corner.

69. At a party of logicians, the host attached either a gold star or a blue star to the back of each of the guests. After all the guests had arrived and mingled with each other, the host announced that there was at least one gold star, and anybody who could prove they had one on their back, without peeking, should claim the prize. No one came forward to claim the prize. Every five minutes after this, the host again announced anybody who could prove they had a gold star should claim the prize, and no one came forward. Finally, after the twentieth time the host asked, everyone rushed forward to claim the prize. How many guests were at the party?

70. *(Trinomial Theorem)* Prove that

$$(a+b+c)^n = \sum_{p+q+r=n} \frac{n!}{p!q!r!} a^p b^q c^r$$

for all $n \in \mathbb{P}$, where the sum is taken over all nonnegative values of p, q, and r for which $p+q+r=n$.

71. What is the maximum number of regions that a plane can be divided into by n straight lines?

72. What is the maximum number of regions that three-dimensional space can be divided into by n planes?
(See the one-hour movie by George Pólya entitled *Let Us Teach Guessing*, distributed by The Mathematical Association of America.)

73. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $0 \le a_i \le 1$ for $1 \le i \le n$. Prove that

$$(1-a_1)(1-a_2)\cdots(1-a_n) \ \ge\ 1-(a_1+a_2+\cdots+a_n).$$

74. A sequence $x_1, x_2, x_3, \ldots$ of real numbers is defined by $x_1 = 1$ and

$$x_{n+1} = \frac{n}{n+1}x_n + 1, \qquad \text{if } n \ge 1.$$

Prove that $x_n = \dfrac{n+1}{2}$ for all positive integers n.

75. A sequence $y_1, y_2, y_3, \ldots$ of integers is defined by $y_1 = 4$, and $y_{n+1} = y_n^2 - 2$, if $n \ge 1$. Prove that

$$y_n = (2+\sqrt{3})^{2^{n-1}} + (2-\sqrt{3})^{2^{n-1}}$$

for all positive integers n.

76. Prove that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$

for all integers $n \ge 2$.

77. Show that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \le 2-\frac{1}{n}$$

for all integers $n \ge 1$.

78–82. *Find the value of each recursive mystery function* $\text{myst}(x_1, x_2, \ldots, x_n)$ *on any n-tuple* $(x_1, x_2, \ldots, x_n)$ *and prove that your value is correct.*

78. $$\text{myst}(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{if } n = 1 \\ x_n - \text{myst}(x_1, x_2, \ldots, x_{n-1}) & \text{if } n > 1 \end{cases}$$

79. $$\text{myst}(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{if } n = 1 \\ x_n \cdot \text{myst}(x_1, x_2, \ldots, x_{n-1}) & \text{if } n > 1 \end{cases}$$

80. $$\text{myst}(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{if } n = 1 \\ x_n & \text{if } x_n > \text{myst}(x_1, x_2, \ldots, x_{n-1}) \\ \text{myst}(x_1, x_2, \ldots, x_{n-1}) & \text{otherwise} \end{cases}$$

81. $$\text{myst}(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{if } n = 1 \\ x_1 - 2\,\text{myst}(x_2, x_3, \ldots, x_n) & \text{if } n > 1 \end{cases}$$

82. $$\text{myst}(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{if } n = 1 \\ \text{myst}(x_1, x_2, \ldots, x_{n-1}) \\ \quad + \text{myst}(x_2, x_3, \ldots, x_n) & \text{if } n > 1 \end{cases}$$

83. Find a recursive definition for the function

$$e(n) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

that gives an approximation to e, the base of natural logarithms.

84. Discuss the following recursive definition of the greatest common divisor for $a \geq 0$ and $b \geq 0$. Is it correct? Is it efficient?

$$\gcd(a, b) = \begin{cases} a & \text{if } b = 0 \\ \gcd(b, a) & \text{if } a < b \\ \gcd(b, a - b) & \text{if } a \geq b > 0 \end{cases}$$