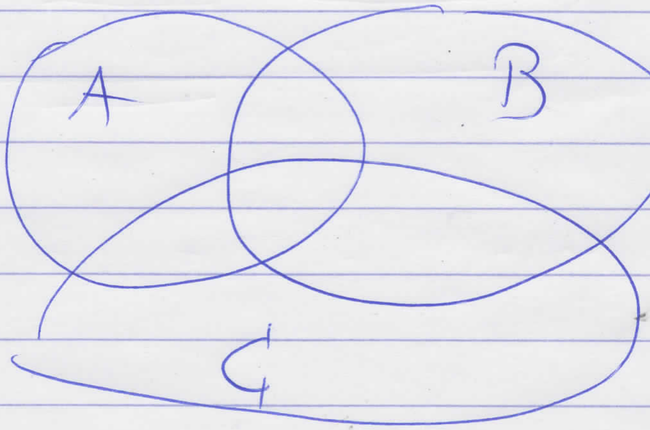
Problem Set 6 page 155 Problem 45:

(*) $\#(A \cup B \cup C) =$
$\#A + \#B + \#C - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$.

Proof:



Let $C \setminus ((C \cap A) \cup (C \cap B))$ be the subset of elements of C which do not belong to $((C \cap A) \cup (C \cap B))$.

$((C \cap A) \cup (C \cap B)) = C \cap (A \cup B)$

Then $A \cup B \cup C$ is the disjoint union of $A \cup B$ and $[C \setminus ((C \cap A) \cup (C \cap B))]$. So

(1) $\#(A \cup B \cup C) = \#(A \cup B) + \#(C \setminus ((C \cap A) \cup (C \cap B)))$, by Theorem 6.62. Similarly,

$\#(C) = \#[C \setminus ((C \cap A) \cup (C \cap B))] + \#((C \cap A) \cup (C \cap B))$, by Theorem 6.62, and so

(2) $\#[C \setminus ((C \cap A) \cup (C \cap B))] = \#(C) - \#((C \cap A) \cup (C \cap B))$.

Now,

(3) $\#((C \cap A) \cup (C \cap B)) = \#(C \cap A) + \#(C \cap B) - \#(C \cap A \cap B)$, by Theorem 6.62.

Equation (*) follows from (1), (2), and (3). Q.E.D

Set 6 page 155 problem 48:

$\mathcal{P}(\emptyset) = \{\emptyset\}$ (one element)

$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$ (two elements)

$\mathcal{P}(\{r,s,t\}) = \{\emptyset, \underbrace{\{r\}, \{s\}, \{t\}, \{r,s\}, \{r,t\}, \{s,t\},}_{6 \text{ elements,}} \{r,s,t\}\}$

Theorem: If $\#X = n$, then $\#(\mathcal{P}(X)) = 2^n$.

Proof by induction: Cases $n=0$ and $n=1$ are checked above.

Induction step: Assume that ~~on~~ the statement is true for every set X with $\#X = n$. Let Y be a set with $\#(Y) = n+1$. Then Y is the disjoint union $X \cup \{y_0\}$, where ~~X~~ y_0 is an element of Y and $X = Y \setminus \{y_0\}$. ~~The~~ Let $\mathcal{S} \subset \mathcal{P}(Y)$ be the subset consisting of subsets of Y which contain y_0. There is a bijection between $\mathcal{P}(X)$ and $\mathcal{S}$ given by

$$f : \mathcal{P}(X) \to \mathcal{S}$$
$$f(\Sigma) = \Sigma \cup \{y_0\}.$$

The inverse of f is $f^{-1}(\tilde{\Sigma}) = \tilde{\Sigma} \setminus \{y_0\}$, so f is invertible and hence bijective. Hence $\#(\mathcal{S}) = \#\mathcal{P}(X)$. Now, $\mathcal{P}(Y)$ is the disjoint union of $\mathcal{P}(X)$ and $\mathcal{S}$, so

$$\#(\mathcal{P}(Y)) = \#(\mathcal{P}(X)) + \#(\mathcal{S}) = 2 \cdot \#(\mathcal{P}(X)) \underset{\text{Induction Hypothesis}}{=} 2 \cdot (2^n) = 2^{n+1}.$$

Q.E.D.