Prob set 3 page 84 #56:

Let p be a prime, Then

$x^2 \equiv y^2 \pmod p$ if and only if

$x \equiv y \pmod p$ OR

$x \equiv -y \pmod p$.

($\Leftarrow$) If $x \equiv y \pmod p$ then $x - y \equiv 0 \pmod p$

If $x \equiv -y \pmod p$ then $x + y \equiv 0 \pmod p$

Hence if $x \equiv y \pmod p$ OR $x \equiv -y \pmod p$

then $(x^2 - y^2) = (x-y)(x+y) \equiv 0 \pmod p$.

We conclude that $x^2 \equiv y^2 \pmod p$.

($\Rightarrow$) If the statement is equivalent to

If $x^2 \equiv y^2 \pmod p$ and $x \not\equiv y \pmod p$, then

$x \equiv -y \pmod p$.

Assume $x^2 \equiv y^2 \pmod p$ and $x \not\equiv y \pmod p$.

Then $p \mid x^2 - y^2 = (x-y)(x+y)$ and

$p \nmid (x-y)$.

Thus, $p \mid (x+y)$, by Theorem 2.53.

So $x \equiv -y \pmod p$.

QED