A strengthening of Euler-Fermat's Theorem

1. Let a positive integer n have the prime decomposition $p_1^{e_1} \cdots p_k^{e_k}$, where the primes p_i are pairwise distinct and the e_i are all positive integers. Set

$$e := \text{lcm}\{\phi(p_1^{e_1}), \cdots, \phi(p_k^{e_k})\} = \text{lcm}\{(p_1^{e_1} - p_1^{e_1-1}), \cdots, (p_k^{e_k} - p_k^{e_k-1})\}.$$

Theorem *If an integer x satisfies* $\gcd(x, n) = 1$*, then* $x^e \equiv 1$ *(mod n).*

Note that e divides $\phi(n) = \prod_{i=1}^{k} \phi(p_i^{e_i})$, and if $\phi(n) = eq$, then $x^{\phi(n)} = (x^e)^q$, so the congruence $x^e \equiv 1 \pmod{n}$ in the above theorem implies the congruence $x^{\phi(n)} \equiv 1 \pmod{n}$ in Euler-Fermat's theorem. In that sense the above theorem is a strengthening of Euler-Fermat's theorem.

Prove the above Theorem using the following steps.

   (a) Prove that $x^{\phi\left(p_i^{e_i}\right)} \equiv 1 \pmod{p_i^{e_i}}$.

   (b) Use the Chinese Remainder Theorem to Prove that if $\phi(p_i^{e_i})$ divides m, for $1 \leq i \leq k$, then $x^m \equiv 1$ (modulo n).

   (c) Prove the statement using the definition of lcm.

2. Use Part 1 to show that if an integer x is not divisible by 3 or 11, then $x^{10} \equiv 1 \pmod{33}$. Note that $\phi(33) = 20$, so the statement does not follow directly from Euler-Fermat's Theorem.

3. Use Part 1 to show that if an odd positive integer x is not divisible by 5, then the last two digit of x^{21} are equal to the last two digits of x.