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73) $a_1, a_2, \ldots \in \mathbb{R}$, $0 \leq a_i \leq 1$ for $1 \leq i \leq n$

Prove:

$$\underbrace{(1-a_1)(1-a_2)\cdots(1-a_n)}_{P(n)} \overset{\circledast}{\geq} 1-(a_1+a_2+\cdots+a_n),$$

Proof: Initial step: $1-a_1 \geq 1-a_1$ ✓

Assume ⊛ holds. Need to show

$$(1-a_1)\cdots(1-a_n)(1-a_{n+1}) \overset{\circledast\circledast}{\geq} 1-(a_1+\cdots+a_n+a_{n+1})$$

$$LHS = P(n)\underbrace{(1-a_{n+1})}_{\geq 0} \underset{\text{Induction hyp}}{\geq}$$

$$\geq (1-(a_1+\cdots+a_n))(1-a_{n+1}) =$$

$$= 1-(a_1+\cdots+a_n) - a_{n+1} + \underbrace{(a_1+\cdots+a_n)a_{n+1}}_{\geq 0}$$

$$\geq 1-(a_1+\cdots+a_{n+1}).$$

Hence the $n+1$ case ⊛⊛ holds. ∎

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$$x_1 = 1, \quad x_{n+1} = \frac{n}{n+1} x_n + 1, \quad \text{if } n \geq 1.$$

Prove that $x_n \underset{(*)}{=} \frac{n+1}{2}$ for all $n \in \mathbb{P}$.

<u>Proof</u>: Initial step $1 = \frac{1+1}{2}$ ✓.

Induction step: Assume $(*)$.

$$x_{n+1} = \frac{n}{n+1} x_n + 1 \underset{\text{Ind Hyp}}{=} \frac{n}{n+1}\left(\frac{n+1}{2}\right) + 1 = \frac{n}{2} + 1 =$$

$$= \frac{n+2}{2} = \frac{(n+1)+1}{2}. \quad \square$$

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$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \underset{(*)}{\leq} 2 - \frac{1}{n} \quad \text{for all } n \geq 1.$$

Proof by Induction.

Initial Step: Case $n=1$: $1 \leq 2 - \frac{1}{1}$ ✓.

Induction Hyp: Assume (*).

Induction Step: Need to show

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \overset{(**)}{\leq} 2 - \frac{1}{n+1}$$

$$\text{LHS} \overset{\text{of } (**)}{\leq} \left(2 - \frac{1}{n}\right) + \frac{1}{(n+1)^2} =$$

By Ind. Hyp.

$$= 2 - \left(\frac{(n+1)^2 - n}{n(n+1)^2}\right) = 2 - \left(\frac{n^2+n+1}{n(n+1)^2}\right)$$

The inequality (**) would follow once we show that

$$\frac{n^2+n+1}{n(n+1)^2} \geq \frac{1}{n+1}$$

$$\Longleftrightarrow$$

$$(n^2+n+1)(n+1) \geq n(n+1)^2$$

$$\Longleftrightarrow$$

$(n^2+n+1) \geq n(n+1)$, which is clear. $\square$