

Fibonacci seq: 1, 1, 2, 3, 5, 8, 13, ...

$$b_1 = 1, b_2 = 1, \quad b_m = b_{m-1} + b_{m-2}, \quad \text{for } m \geq 3$$

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Prove that  $b_{m+1} < \left(\frac{7}{4}\right)^m$

Proof by strong Induction:

Initial step: checks for  $m=1, m=2,$

Let  $m \geq 2$ . Assume that  $b_k < \left(\frac{7}{4}\right)^k$  for  $1 \leq k \leq m$ .

By Induction Hyp.

$$b_{m+1} \stackrel{\text{def}}{=} b_m + b_{m-1} < \left(\frac{7}{4}\right)^m + \left(\frac{7}{4}\right)^{m-1} = \left(\frac{7}{4}\right)^{m-1} \underbrace{\left(\frac{7}{4} + 1\right)}_{\frac{11}{4}}$$

$$\frac{11}{4} < \left(\frac{7}{4}\right)^2 = \frac{49}{16}. \quad \text{Hence,}$$

$$b_{m+1} < \left(\frac{7}{4}\right)^{m+1}$$

63)  $a = \frac{1+\sqrt{5}}{2}$   $b = \frac{1-\sqrt{5}}{2}$   $(a+b=1, ab=-1)$   
 $a-b=\sqrt{5}$

Prove that  $f_m = \frac{a^m - b^m}{\sqrt{5}}$  for all  $m$

Step 1:  $f_m = a^{m-1} + a^{m-2}b + \dots + b^{m-1}$  for  $m \geq 2$

Proof by induction:

$f_2 = a^1 + b^1 = 1$  ✓

$f_3 = a^2 + ab + b^2 = (a+b)^2 - ab = 1 - (-1) = 2$  ✓

Assume  $m \geq 3$  and  $f_k = a^{k-1} + a^{k-2}b + \dots + b^{k-1}$  for  $2 \leq k \leq m$ .

$f_{m+1} \stackrel{\text{def}}{=} f_m + f_{m-1} \stackrel{\text{Ind Hyp}}{=} (a^{m-1} + a^{m-2}b + \dots + b^{m-2}a + b^{m-1})(a+b) + a^{m-2} + a^{m-3}b + \dots + b^{m-2}$

Need to show

$f_{m+1} \stackrel{(*)}{=} a^m + a^{m-1}b + \dots + b^m$

$a^m + a^{m-1}b + \dots + b^{m-1}a + a^{m-2} + a^{m-3}b + \dots + b^{m-2} + b^m$

$\left. \begin{matrix} + a^{m-1}b + \dots + b^{m-1}a + b^m \\ + a^{m-2} + a^{m-3}b + \dots + b^{m-2} \end{matrix} \right\} = b^m$

Hence  $(*)$  holds.

since  $ab = -1$



Step 2! We need to show that  $(a-b)^m = a^m - b^m$  for  $m \geq 2$ .

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + \dots + b^{m-1}, \text{ for } m \geq 2$$

Indeed,  $\frac{a^m - b^m}{a - b} = \dots$  □

Proof by induction:

$$b_1 = a^1 + b^1 = 1$$

$$b_2 = a^2 + ab + b^2 = (a+b)^2 - ab = 1 - (-1) = 2$$

Assume  $m \geq 3$  and  $b_k = a^{k-1} + a^{k-2}b + \dots + b^{k-1}$  for  $2 \leq k \leq m$ .

Ind. Hyp

$$b_{m+1} = b_m + b_{m-1} = (a^{m-1} + a^{m-2}b + \dots + b^{m-1}) + (a^{m-2} + a^{m-3}b + \dots + b^{m-2})$$

Need to show

$$b_{m+1} = a^m + a^{m-1}b + \dots + b^m$$

$$\left. \begin{aligned} & a^m + a^{m-1}b + \dots + b^{m-1}a + \\ & + a^{m-2}b + \dots + b^{m-2}a + b^m \end{aligned} \right\} = b^m$$

Hence (a) holds. □ since  $ab = -1$