Fibonacci seq: 1, 1, 2, 3, 5, 8, 13, ...

$f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$

Prob 62 Set 4 page 108:

Prove that $f_{n+1} < \left(\frac{7}{4}\right)^n$

Proof by strong Induction:

Initial step: checks for $n = 1$, $n = 2$.

Let $n \geq 2$. Assume that $f_k < \left(\frac{7}{4}\right)^k$ for $1 \leq k \leq n$.

$$f_{n+1} \overset{def}{=} f_n + f_{n-1} \overset{\text{By Induction Hyp.}}{<} \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} = \left(\frac{7}{4}\right)^{n-1} \underbrace{\left(\frac{7}{4} + 1\right)}_{\frac{11}{4}}$$

$\frac{11}{4} < \left(\frac{7}{4}\right)^2 = \frac{49}{16}$. Hence,

$f_{n+1} < \left(\frac{7}{4}\right)^{n+1}$.

Ch 4 page 108

63) $a = \dfrac{1+\sqrt{5}}{2}$ $b = \dfrac{1-\sqrt{5}}{2}$ $\left(a+b=1,\ ab=-1,\ a-b=\sqrt{5}\right)$

Prove that $f_n = \dfrac{a^n - b^n}{\sqrt{5}}$ for all n

Step 1: $f_n = a^{n-1} + a^{n-2}b + \cdots + b^{n-1}$ for $n \geq 2$

Proof by induction:

$f_2 = a^1 + b^1 = 1$ ✓

$f_3 = a^2 + \underbrace{ab}_{-1} + b^2 = (a+b)^2 - ab = 1^2 - (-1) = 2$ ✓

Assume $n \geq 3$ and $f_k = a^{k-1} + a^{k-2}b + \cdots + b^{k-1}$ for $2 \leq k \leq n$.

$$f_{n+1} \overset{def}{=} f_n + f_{n-1} \overset{\text{Ind Hyp}}{=}$$

$$\left(a^{n-1} + a^{n-2}b + \cdots + b^{n-2}a + b^{n-1}\right)\overbrace{(a+b)}^{1}$$

$$+ \quad a^{n-2} + a^{n-3}b + b^{n-2}$$

Need to show

$$f_{n+1} \overset{(*)}{=} a^n + a^{n-1}b + \qquad + b^n$$

$$a^n + a^{n-1}b + \qquad + b^{n-1}a +$$

$$+ a^{n-1}b + \qquad b^{n-1}a + b^n + \Big\} = b^n$$

$$a^{n-2} + \qquad + b^{n-2}$$

since $\boxed{ab=-1}$

Hence (*) holds.

Step 2: We need to show that

$$\frac{a^n - b^n}{\sqrt{5}} = a^{n-1} + a^{n-2}b + \dots + b^{n-1}, \text{ for } n \geq 2.$$

Indeed, $\frac{a^n - b^n}{a-b} =$ // .

$\square$