Exercise Set 1

1–6. *Determine which of the following sentences are statements. What are the truth values of those that are statements?*

1. $7 > 5$
2. $5 > 7$
3. Is $5 > 7$?
4. $\sqrt{2}$ is an integer.
5. Show that $\sqrt{2}$ is not an integer.
6. If 5 is even then $6 = 7$.

7–12. *Write the truth tables for each expression.*

7. NOT(NOT P)
8. NOT(P OR Q)
9. $P \Longrightarrow (Q$ OR $R)$
10. (P AND Q) $\Longrightarrow R$
11. (P OR NOT Q) $\Longrightarrow R$
12. NOT $P \Longrightarrow (Q \Longleftrightarrow R)$

13. P UNLESS Q is defined as (NOT Q) $\Longrightarrow P$. Show that this statement has the same truth table as P OR Q. Give an example in common English showing the equivalence of P UNLESS Q and P OR Q.

14. Write down the truth table for the *exclusive or* connective XOR, where the statement P XOR Q means (P OR Q) AND NOT (P AND Q). Show that this is equivalent to NOT($P \Longleftrightarrow Q$).

15. Write down the truth table for the *not or* connective NOR, where the statement P NOR Q means NOT(P OR Q).

16. Write down the truth table for the *not and* connective NAND, where the statement P NAND Q means NOT(P AND Q).
[Electronic circuits often use NOR, NAND, and XOR gates.]

17–21. *Write each statement using P, Q, and connectives.*

17. P whenever Q.
18. P is necessary for Q.
19. P is sufficient for Q.
20. P only if Q.
21. P is necessary and sufficient for Q.

22. Show that the statements NOT (P OR Q) and (NOT P) AND (NOT Q) have the same truth tables, and give an example of the equivalence of these statements in everyday language.

23. Show that the statements P AND (Q AND R) and (P AND Q) AND R have the same truth tables. This is the *associative law* for AND.

24. Show that P AND (Q OR R) and (P AND Q) OR (P AND R) are statements with the same truth tables. This is a *distributive law.*

25. Is (P AND Q) $\Longrightarrow R$ equivalent to $P \Longrightarrow (Q \Longrightarrow R)$? Give reasons.

26–28. *Let P be the statement* "It is snowing" *and let Q be the statement* "It is freezing." *Write each statement using P, Q, and connectives.*

26. If it is snowing, then it is freezing.
27. It is freezing but not snowing.
28. When it is not freezing, it is not snowing.

29–32. *Let* P *be the statement* "I can walk," Q *be the statement* "I have broken my leg," *and* R *be the statement* "I take the bus." *Express each statement as an English sentence.*

29. $Q \Longrightarrow \text{NOT } P$

30. $P \Longleftrightarrow \text{NOT } Q$

31. $R \Longrightarrow (Q \text{ OR NOT } P)$

32. $R \Longrightarrow (Q \Longleftrightarrow \text{NOT } P)$

33–39. *Express each statement as a logical expression using quantifiers. State the universe of discourse.*

33. There is a smallest positive integer.
34. There is no smallest positive real number.
35. Every integer is the product of two integers.
36. Every pair of integers has a common divisor.
37. There is a real number x such that, for every real number y, $x^3 + x = y$.
38. For every real number y, there is a real number x such that $x^3 + x = y$.
39. The equation $x^2 - 2y^2 = 3$ has an integer solution.

40. Express the following quote due to Abraham Lincoln as a logical expression using quantifiers: "You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time."

41–44. *Negate each expression, and simplify your answer.*

41. $\forall x,\ (P(x) \text{ OR } Q(x))$

42. $\forall x,\ ((P(x) \text{ AND } Q(x)) \Longrightarrow R(x))$

43. $\exists x,\ (P(x) \Longrightarrow Q(x))$

44. $\exists x,\ \forall y\ (P(x) \text{ AND } Q(y))$

45–50. *If the universe of discourse is the real numbers, what does each statement mean in English? Are they true or false?*

45. $\forall x\ \forall y,\ \ (x \geq y)$

46. $\exists x\ \exists y,\ \ (x \geq y)$

47. $\exists y\ \forall x,\ \ (x \geq y)$

48. $\forall x\ \exists y,\ \ (x \geq y)$

49. $\forall x\ \exists y,\ \ (x^2 + y^2 = 1)$

50. $\exists y\ \forall x,\ \ (x^2 + y^2 = 1)$

51–54. *Determine whether each pair of statements is equivalent. Give reasons.*

51. $\exists x,\ (P(x) \text{ OR } Q(x))$ $\qquad (\exists x,\ P(x)) \text{ OR } (\exists x,\ Q(x))$
52. $\exists x,\ (P(x) \text{ AND } Q(x))$ $\qquad (\exists x,\ P(x)) \text{ AND } (\exists x,\ Q(x))$
53. $\forall x,\ (P(x) \Longrightarrow Q(x))$ $\qquad (\forall x,\ P(x)) \Longrightarrow (\forall x,\ Q(x))$
54. $\forall x,\ (P(x) \text{ OR } Q(y))$ $\qquad (\forall x,\ P(x)) \text{ OR } Q(y)$

55–61. *Write the contrapositive and the converse of each statement.*

55. If Tom goes to the party, then I will go to the party.
56. If I do my assignments, then I get a good mark in the course.
57. If $x > 3$, then $x^2 > 9$.
58. If $x < -3$, then $x^2 > 9$.
59. If an integer is divisible by 2, then it is not prime.
60. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.
61. If $x^2 + y^2 = 9$, then $-3 \leq x \leq 3$.

62. Let S and T be sets. Prove that if $x \notin S \cap T$, then $x \notin S$ or $x \notin T$.
63. Let a and b be real numbers. Prove that if $ab = 0$, then $a = 0$ or $b = 0$.
64. Use the Contrapositive Proof Method to prove that

$$(S \cap T = \emptyset) \text{ AND } (S \cup T = T) \Longrightarrow S = \emptyset.$$

65–70. *Prove or give a counterexample to each statement.*

65. $\forall x \in \mathbb{R},\ (x^2 + 5x + 7 > 0)$
66. If m and n are integers with mn odd, then m and n are odd.
67. If x and y are real numbers, then $\forall x, \exists y\ (x^2 > y^2)$.
68. $(S \cap T) \cup U = S \cap (T \cup U)$, for any sets S, T, and U.
69. $S \cup T = T \Longleftrightarrow S \subseteq T$
70. If x is a real number such that $x^4 + 2x^2 - 2x < 0$, then $0 < x < 1$.

71. Prove the *distributive law* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
72. Prove the *distributive law* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

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73. If S, T, and U are sets, the statement $S \cap T \subseteq U$ can be expressed as

$$\forall x,\ \ ((x \in S \text{ AND } x \in T) \Longrightarrow x \in U).$$

Express and simplify the negation of this expression, namely $S \cap T \not\subseteq U$, in terms of quantifiers.

74. If S and T are sets, the statement $S = T$ can be expressed as

$$\forall x,\ \ (x \in S \Longleftrightarrow x \in T).$$

What does $S \neq T$ mean? How would you go about showing that two sets are not the same?

75. The definition of the limit of a function, $\lim_{x \to a} f(x) = L$, can be expressed using quantifiers as

$$\forall \epsilon > 0\ \ \exists \delta > 0\ \ \forall x,\ (0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon).$$

Use quantifiers to express the negation of this statement, which would be a definition of $\lim_{x \to a} f(x) \neq L$.

76. Use truth tables to show that the statement $P \Longrightarrow (Q \text{ OR } R)$ is equivalent to the statement $(P \text{ AND NOT } Q) \Longrightarrow R$.
[This explains the Proof Method 1.56 for $P \Longrightarrow (Q \text{ OR } R)$.]

77. Use truth tables to show that the statement $(P \text{ OR } Q) \Longrightarrow R$ is equivalent to the statement $(P \Longrightarrow R) \text{ AND } (Q \Longrightarrow R)$.
[This explains the Proof Method 1.57 for $(P \text{ OR } Q) \Longrightarrow R$.]

78. Use truth tables to show that the statement $P \Longrightarrow (Q \text{ AND } R)$ is equivalent to the statement $(P \Longrightarrow Q) \text{ AND } (P \Longrightarrow R)$.
[This explains the Proof Method 1.58 for $P \Longrightarrow (Q \text{ AND } R)$.]

79. Is the statement $(P \text{ AND } Q) \Longrightarrow R$ equivalent to $(P \Longrightarrow R) \text{ OR } (Q \Longrightarrow R)$? Give reasons.

80. Is the statement $P \Longrightarrow (Q \Longrightarrow R)$ equivalent to $(P \Longrightarrow Q) \Longrightarrow R$? Give reasons.

81. Show that the statement $P \text{ OR } Q \text{ OR } R$ is equivalent to the statement $(\text{ NOT } P \text{ AND } \text{ NOT } Q) \Longrightarrow R$.

82–83. *For each truth table, find a statement involving P and Q and the connectives,* AND, OR, *and* NOT *that yields that truth table.*

82.

P	Q	???
T	T	T
T	F	T
F	T	F
F	F	T

83.

P	Q	???
T	T	F
T	F	T
F	T	F
F	F	F

84. **(a)** How many nonequivalent statements are there involving P and Q?
(b) How many nonequivalent statements are there involving $P_1, P_2, \ldots, P_n$?