Math 300.1 Quiz 4 Fall 2017

Name: Solution

We say that two sets $A$ and $B$ have the same cardinality, and write $\sharp A = \sharp B$, if there exists a bijective map $f : A \to B$. Let $A$, $B$, and $C$ be sets.

1. Prove that $\sharp A = \sharp A$.

Let $f = 1_A : A \to A$ be the identity map, $f(x) = x$ $\forall x \in A$. Then $f \circ f = 1_A$ so $f$ is invertible, hence bijective, by the Inversion Theorem. Hence $\# A = \# A$.

2. Prove that if $\sharp A = \sharp B$, then $\sharp B = \sharp A$. Assume $\# A = \# B$. Then there exists a bijective map $f : A \to B$. By the Inversion Theorem, $f$ is invertible. Now $f^{-1} : B \to A$ is invertible as well, $(f^{-1})^{-1} = f$, by definition of the inverse function. Hence, $f^{-1}$ is bijective, by the Inversion Theorem, and so $\# B = \# A$.

3. Prove that if $\sharp A = \sharp B$ and $\sharp B = \sharp C$, then $\sharp A = \sharp C$.

Assume that $\# A = \# B$ and $\# B = \# C$. Then there exist bijective maps $f : A \to B$ and $g : B \to C$. The composition of bijective maps is bijective. Hence, $(g \circ f) : A \to C$ is bijective and $\# A = \# C$.