Math 235 Final Exam Spring 1999

1. (15 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   a) What is the rank of A?

   b) Find a basis for the null space $Null(A)$ of A.

   c) Find a basis for the column space of A.

   d) Find a basis for the row space of A.

2. (4 points) The null space of the 5×6 matrix A is 2 dimensional. What is the dimension of (a) the Row space of A? (b) the Column space of A? **Justify your answer!**

3. (15 points)

   (a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 4 & -4 & -1 \end{pmatrix}$ is $-(\lambda-1)(\lambda+1)(\lambda-2)$.

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   (c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

4. (12 points) Determine for which of the following matrices A below there exists an invertible matrix P (with real entries) such that $P^{-1}AP$ is a diagonal matrix. You do **not** need to find P. **Justify your answer!**

   (a) $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

5. (22 points) Let W be the plane in $\mathbb{R}^3$ spanned by $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

   Note: Parts 5a, 5b, 5c are mutually independent and are not needed for doing parts 5d, 5e, 5f.

(a) Find the length of v_1.

(b) Find the distance between the two points v_1 and v_2 in $\mathbb{R}^3$.

(c) Find a vector of length 1 which is orthogonal to W.

(d) Find the projection of v_2 to the line spanned by v_1.

(e) Write v_2 as the sum of a vector parallel to v_1 and a vector orthogonal to v_1.

(f) Find an orthogonal basis for W.

6. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(a) Find the projection of $b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ to W.

(b) Find the distance from b to W.

(c) Find a least square solution to the equation $Ax = b$ where A is the 3×2 matrix with columns u_1 and u_2. I.e., find a vector x in $\mathbb{R}^2$ which minimizes the length $\|Ax - b\|$.

(d) Find the coefficients c_0, c_1 of the line $y(x) = c_0 + c_1 x$ which best fits the three points $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (0, 2)$, $(x_3, y_3) = (1, 1)$ in the x, y plane. The line should minimize the sum $\sum_{i=1}^{3} [y(x_i) - y_i]^2$. **Justify your answer!**

7. (16 points) The vectors $v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .8 & .5 \\ .2 & .5 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______

The eigenvalue of v_2 is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ approaches _____. Justify your answer.