Math 235 Midterm 2 Spring 2014

1. (18 points) You are given below the matrix A together with its row reduced echelon form C

$$A = \begin{pmatrix} 1 & -1 & -3 & -3 & 0 & -3 \\ 1 & 0 & 2 & 3 & 0 & 4 \\ 2 & 0 & 4 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 1 & 5 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: you do **not** have to check that A and C are indeed row equivalent.

a) Determine the rank of A. Explain how it is determined by the matrix C.

b) Find a basis for the kernel $\ker(A)$ of A. Justify your answer!

c) Find a basis for the image $\mathrm{im}(A)$ of A. Justify your answer!

d) Let $\mathcal{B}$ be the basis you found in part 1c for the image of A and let $\vec{a}_6$ be the sixth column of A. Find the $\mathcal{B}$-coordinate vector $[\vec{a}_6]_{\mathcal{B}}$ of $\vec{a}_6$.

2. (12 points) For which values of the constant k do the vectors below form a basis of $\mathbb{R}^3$. Justify your answer!

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \\ k \end{pmatrix}.$$

3. (16 points) Let $\vec{v}_1$ be a non-zero vector in $\mathbb{R}^2$. Recall that the reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to the line spanned by $\vec{v}_1$, is given by

$$T(\vec{x}) = 2\left(\frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \vec{x}. \tag{1}$$

(a) Let $\mathcal{B} := \{\vec{v}_1, \vec{v}_2\}$ be a basis of $\mathbb{R}^2$ such that $\vec{v}_1 \cdot \vec{v}_2 = 0$ (the two vectors are orthogonal). Let T be the reflection with respect to the line spanned by $\vec{v}_1$. Express $T(\vec{v}_1)$ and $T(\vec{v}_2)$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

(b) Use your calculations in part **??** to find the $\mathcal{B}$-matrix B of T.

(c) Assume from now on that T is the reflection with respect to the line spanned by $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(d) Use your work in part **??** to show that the matrix of T with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ is $A = \frac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$.

(e) Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be the basis of $\mathbb{R}^2$, where the vector $\vec{v}_1$ is given in part **??** and $\vec{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Note that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal $\vec{v}_1 \cdot \vec{v}_2 = 0$. Find a matrix S, such that $S^{-1}AS$ is equal to the $\mathcal{B}$-matrix B of T you found in part **??**, where A is the standard matrix you found in part **??**.

(f) Explicitly verify that the matrices A, B, and S in part **??** satisfy the equality $SB = AS$, by calculating each side.

4. (12 points) Let A be a 5×4 matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$. We are given that the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ belongs to the kernel of A and the vectors $\begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 0 \end{pmatrix}$ span the image of A.

   (a) Express $\vec{a}_4$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

   (b) Determine the dimension of the image of A. Justify your answer.

   (c) Determine the dimension of the kernel of A. Justify your answer.

5. (14 points) Let P_2 be the space of all polynomials $a_0 + a_1 t + a_2 t^2$ of degree ≤ 2. Find a basis for the subspace W of P_2 consisting of all polynomials $f(t)$ satisfying $f'(1) = 0$. Explain why the set you found spans W and why it is linearly independent.

6. (14 points) Determine which of the following subsets is a subspace by verifying the properties in the definition of a subspace or by showing that one of those properties does not hold.

   (a) The subset W of all 2×2 matrices A satisfying $AB = BA$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   (b) The subset W of $\mathbb{R}^4$ consisting of vectors of the form $\begin{pmatrix} x - y \\ y - z \\ x + z \\ y \end{pmatrix}$, where x, y, z are arbitrary real numbers.

7. (14 points)

   (a) Consider a matrix A and let B be the row reduced echelon form of A. Explain why the statement is true or provide a counter example.

      i. Is $\ker(A)$ necessarily equal to $\ker(B)$?

      ii. Is the image of A necessarily equal to the image of B?

   (b) Let A be a 4×3 matrix and B a 3×4 matrix. Show that $\text{rank}(AB) \leq 3$. Hint: Relate $\text{im}(AB)$ and $\text{im}(A)$?