DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 235 **Final Exam** **Fall 2010**

Instructions:

- **Please use correct notation when writing matrices and vectors.**
- **In True-or-False questions, please give reasoning or a counter-example. Examples alone are not reasons.**
- **Explain how you arrived at your answers, please, and show your algebraic calculations. Use the back of the preceding page if necessary.**
- **Please justify your statements. Unsubstantiated answers receive no credit.**
- bf If L is a linear map from a vector space V to V and B is a basis of V, then we denote the matrix of L with respect to the basis B as $[L]_{BB}$. In some sections slightly different notations were used for this.

1: True or False. (Please support your answer with a brief reason or a counter-example.)
1a: Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear. Assume that for all $y \in \mathbb{R}^n$ the equation $Lx = y$ has a solution. Then the solution is unique.
1b: Let P_n denote the vector space of polynomials of degree less than or equal to n. Let $F : P_3 \to P_2$ be linear. Then $F(p) = 2 - 3t^2$ has a solution.
1c: Assume that $S = \{u_1, u_2, u_3\}$ is a set of three non-zero orthogonal vectors in $\mathbb{R}^3$, so $< u_i, u_j >= 0$ if $i \neq j$. Then S is a basis of $\mathbb{R}^3$.
1d: Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear and that x_1, x_2 are both solutions to the equation $Lx = y$. Then $x_1 - x_2$ is in the kernel of L.
1e: Suppose that P is the vector space of all polynomials in the variable t. Let

$$T : P \to P, \quad p(t) \mapsto \frac{d^2p}{dt^2} - 2\frac{dp}{dt} - 3p + 7.$$

Then T is linear.

2: Let P_2 be the vector space of polynomials of degree less than or equal to 2 with basis $S = \{1, t, t^2\}$. Let $T : P_2 \to P_2 : p(t) \mapsto 3p''(t) - 2p'(t) + p(t)$.

2a: Verify that T is a linear map.
2b: Compute the matrix of T with respect to the basis S.
2c: Is T an isomorphism? Why?
2d: Find all polynomials $p(t)$ so that $T(p(t)) = 2 - 3t^2$.

3: Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis, and let $B = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ be another basis of $\mathbb{R}^2$.

3a: Suppose $v \in \mathbb{R}^2$ has coordinates $[v]_S = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with respect to the standard basis S. What are its coordinates $[v]_B$ with respect to B?

3b: If a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix

$$[L]_{SS} = \begin{pmatrix} 9 & -8 \\ 10 & -9 \end{pmatrix}$$

in the standard basis S, what is its matrix $[L]_{BB}$ in the basis B?

4: Compute the determinant of the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & 3 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Find $det(B^{23})$. Find $det(B^{-1})$ if B is invertible.

5: Let $X = \begin{pmatrix} 1 & -4 \\ -16 & -11 \end{pmatrix}$.

5a: Compute the characteristic polynomial of X.

5b: Find the eigenvalues of X.

5c: Find the eigenvectors for each eigenvalue.

5d: Is X diagonalizable,that is, can we find a matrix so that AXA^{-1} is a diagonal matrix? If so find the matrix A. If not explain why not.

6: Let $M = \begin{pmatrix} 5 & -5 \\ 2 & -1 \end{pmatrix}$.

6a: Compute the characteristic polynomial of M.

6b: Find the eigenvalues and eigenvectors of M.

6c: The eigenvalues are not real. Find a basis B of $\mathbb{R}^2$ so that the matrix of M with respect to the basis B is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

7: Let P_2 denote the vector space of polynomials of degree less than or equal to 2. Is $S = \{1, 1-t, 1-t^2\}$ a basis of P_2? Why? (Note that explaining why is the important part of the question.)