1: True or False. (Please support your answer with a brief reason or a counter-example.)

1a: Let M be an $n \times n$ matrix. If the columns of M are independent, then the kernel of M is just the zero vector.

TRUE. $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \ker M \iff x_1 M_1 + \cdots + x_n M_n = 0$ vector

where $M_1, \ldots M_n$ are columns of M. This says there is a non-trivial linear relation $\iff$ there is a non-zero element in $\ker(M)$

1b: If the set of vectors $\{u, v, w\}$ is independent, then w must be a linear combination of u and v.

False. If $w = au + bv$, then $au + bv - 1 \cdot w = 0$ is a non-trivial linear relation among (u, v, w). But since $\{u, v, w\}$ is independent, there are no such relations.

1c: The image of a 3×4 matrix M is a subspace of $\mathbb{R}^3$.

$\overset{\leftarrow 4 \rightarrow}{3 \updownarrow \boxed{\ M.\ }}$ $M: \mathbb{R}^4 \to \mathbb{R}^3$. The image of a matrix is a subspace

TRUE

1d: The function

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

is linear. True. (a) $T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - y_1 - y_2 \\ x_1 + x_2 + y_1 + y_2 \end{pmatrix}$ These

(b) $T\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$ $\quad T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \end{pmatrix} + \begin{pmatrix} x_2 - y_2 \\ x_2 + y_2 \end{pmatrix}$ are equal

$= \begin{pmatrix} \lambda x - \lambda y \\ \lambda x + \lambda y \end{pmatrix} = \lambda T\begin{pmatrix} x \\ y \end{pmatrix}$.

1e: The set of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ so that $x^2 + y^2 + z^2 = 1$ is a subspace of $\mathbb{R}^3$.

False. $(1, 0, 0)$ is an element of this subset, but $2(1, 0, 0)$ is not.

2: Consider the following system of equations:

$$
\begin{aligned}
-x - z - 2w &= 0 \\
2x + y + 3z + 5w &= 3 \\
-x + y - w &= 3
\end{aligned}
$$

2a: Express this as a matrix equation $AX = B$ with

$$A = \begin{pmatrix} -1 & 0 & -1 & -2 \\ 2 & 1 & 3 & 5 \\ -1 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

2b: Find all solutions $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ using row reduction (Gaussian elimination).

$$\left(\begin{array}{cccc|c} -1 & 0 & -1 & 2 & 0 \\ 2 & 1 & 3 & 5 & 3 \\ -1 & 1 & 0 & -1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{ \begin{aligned} x + z + 2w &= 0 \\ y + z + w &= 3 \end{aligned} \right\}$$

$x = -z - 2w$

$y = 3 - z - w$

$w =$ anything

$z =$ anything.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

3: Find an equation that a, b, c, d must satisfy so that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$$

is in the span of the set of vectors

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ -1 & 1 & b \\ 2 & 0 & c \\ 3 & 1 & d \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 2 & a+b \\ 0 & -2 & c-2a \\ 0 & -2 & d-3a \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 2 & a+b \\ 0 & 0 & c-2a+a+b \\ 0 & 0 & d-3a+a+b \end{array}\right) \begin{array}{c} \to \\ \Rightarrow \end{array}$$

$$\left.\begin{array}{l} \to \ c-a+b=0 \\ \Rightarrow \ d-2a+b=0 \end{array}\right\} - \text{both equations}$$

4a: Let $F : \mathbb{R}^n \to \mathbb{R}^m$. Define what it means for F to be linear.

$$F(u+v) = F(u) + F(v) \quad u, v \in \mathbb{R}^n$$
$$F(\lambda u) = \lambda F(u) \quad \lambda \in \mathbb{R}, u \in \mathbb{R}^n$$

4b: Suppose a linear map $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & -2 & -1 & -3 \\ 2 & 1 & 3 & 1 \end{pmatrix}.$$

Find a basis for $\mathrm{im}(F)$ and compute the rank of F.

Row reduce: $\begin{array}{c} \begin{matrix} * & * & + & * \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$ Basis image is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 1 \end{pmatrix} \right\}$

rank of F is 3

4c: Let F be the map in 4b. Find a basis for $\ker(F)$ and compute its dimension (the nullity of F).

dim of kernel is 1.

$$\begin{aligned} x &= -z \\ y &= -z \\ z &= z \\ w &= 0 \end{aligned}$$

so basis is $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

5: The image of a matrix M of size 5×5 has dimension 2.

5a: How many independent columns vectors does M have? 2.

5b: What is the dimension of the kernel of M? 3

5c: Does the equation $MX = b$ have a solution for every $b \in \mathbb{R}^5$? Why? no

im M has dim 2 which is strictly less than dim of target of M ($\mathbb{R}^5$ has dim 5)

6a: Find a basis for the subspace $V \subset \mathbb{R}^4$ consisting of all vectors orthogonal to $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & 1 \end{pmatrix}.$$

Row reduced: $\begin{pmatrix} 1 & 0 & 1 & 2/3 \\ 0 & 1 & 0 & 1/3 \end{pmatrix}. \rightarrow$

$$\begin{aligned} x &= -z - 2/3\, w \\ y &= -1/3\, w \\ z &= z \\ w &= w \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -2/3 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$$ so $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of V.

6b: What is the dimension of this subspace V in part 6a? $\dim V = 2$

7: Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map. Let

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 4 \\ 0 \\ 4 \end{pmatrix}.$$

Assume that

$$f(u) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and that $f(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. What is $f(u+v)$? Explain why.

$$f(u+v) = f(u) + f(v) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$