DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 235 SPRING 2011
EXAM 2

Your Name: ______________________________

Student ID: ______________________________

Your Instructor's Name: ______________________________

This is a two hours exam. This exam paper consists of 6 questions. It has 10 pages, where the last is a blank page.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (18)	________
2. (12)	________
3. (18)	________
4. (16)	________
5. (18)	________
6. (18)	________
TOTAL (100)	

1. (18 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 1 & 4 \\ 2 & 0 & 4 & 3 & 5 & 2 \\ 3 & 2 & 4 & 6 & 9 & 10 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do **not** need to check that A and B are indeed row equivalent.

(a) Find a basis for the kernel $\ker(A)$ of A.

(b) Find a basis for the image $im(A)$ of A.

(c) Does the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ belong to the image of A? Use part 1b to minimize your computations. Justify your answer!

2. (12 points)

   (a) Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation. What are the possible values of $\dim(\ker(T))$? Justify your answer!

(b) Let A and B be $n \times n$ matrices. Assume that $AB = 0$. Show that the image of B is contained in the kernel of A.

(c) Let A and B be $n \times n$ matrices and assume that the image of B is contained in the kernel of A. Show that $\text{rank}(B) \leq \dim(\ker(A))$. Explain why it follows that $\text{rank}(A) + \text{rank}(B) \leq n$.

3. (18 points) Let $\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

   (a) Show that $\{\vec{v}_2, \vec{v}_3\}$ form a basis for the subspace P of $\mathbb{R}^3$ orthogonal to $\vec{v}_1$.

   (b) Consider the basis $\beta := \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_1$. Find the β-matrix B of T (the matrix of T in the basis β). Justify your answer!

(c) Let S be the 3×3 matrix $(\vec{v}_1 \vec{v}_2 \vec{v}_3)$ with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$. Express the standard matrix A of T in terms of the matrix S and the matrix B you found in part 3b. (You do not need to simplify your answer).

4. (18 points) Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices and P an invertible 2×2 matrix. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the function sending a matrix M to $T(M) = P^{-1}MP$.

(a) Show that T is a linear transformation.

(b) Show that T is an isomorphism by explicitly finding T^{-1}. Carefully justify your answer!

(c) Assume now that $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find the matrix B of T in part 4a in the basis $\beta := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{2\times 2}$.

5. (16 points) Let P_2 be the vector space of polynomials of degree ≤ 2. (i) Which of the following subsets W of P_2 are subspaces? In each case verify the three conditions in the definition of a subspace, or demonstrate that one of them is violated.
(ii) Find a basis for those that are subspaces.

   (a) $W = \{f(t) \ : \ f'(0) = 1\}$ is the subset of polynomial functions $f(t)$, such that the value of its derivative at $t = 0$ is 1.

   (b) $W = \{f(t) \ : \ f(1) = f'(2)\}$.

6. (18 points) Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation given by $T(f(t)) = \begin{bmatrix} f'(0) \\ f(1) \\ f(-1) \end{bmatrix}$.

   The first entry on the right hand side above is the value of the *derivative* f' at 0.

   (a) Find a basis (consisting of *polynomials*) for the kernel $\ker(T)$. Carefully justify why the set you found is a basis.

   (b) Use your answer in part 6a in order to determine the rank and nullity of T. Justify your answer!

   (c) Find a basis for the image $im(T)$. Justify your answer!

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