Math 235 section 4 Midterm 1 Spring 2009

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + 3x_5 &= 1 \\ 2x_1 + x_2 + x_4 + 4x_5 &= 1 \\ x_1 - x_3 + x_4 + 2x_5 &= 0 \end{aligned}$$

is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$. Use at most seven elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

   b) Find the general solution for the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

2. a) (8 points) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$.

   b) (2 points) Use matrix multiplication to check that the matrix you found is indeed A^{-1}.

   c) (5 points) Let A, B, C be $n \times n$ matrices, with A and B invertible, which satisfy the equation $ABCB^{-1} - B = A$. Express C in terms of A and B. Show all your work.

3. (18 points) Recall that two $n \times n$ matrices A and B are said to commute, if $AB = BA$.

   (a) Find all 2×2 matrices, which commute with the matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$.

   (b) Let A and B be two $n \times n$ matrices. Show that if A commutes with B and B is invertible, then A commutes with B^{-1}.

4. (17 points) Let A be an $m \times n$ matrix, $\vec{b}$ a non-zero vector in $\mathbb{R}^n$, $\vec{x}_1$ a solution of the equation $A\vec{x} = \vec{b}$, and $\vec{x}_h$ a solution of the equation $A\vec{x} = \vec{0}$.

   (a) Show that $\vec{x}_1 + \vec{x}_h$ is a solution of the equation $A\vec{x} = \vec{b}$.

   (b) Let $\vec{x}_2$ be another solution of the system $A\vec{x} = \vec{b}$. Show that $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

   (c) Let A be the 2×2 matrix of the projection of $\mathbb{R}^2$ onto a line L through the origin and a non-zero vector $\vec{b}$. Let $\vec{u}$ be a unit vector orthogonal to L. Draw a picture describing geometrically the set of solutions $\vec{x}$ of the system $A\vec{x} = \vec{b}$, in terms of $\vec{u}$ and $\vec{b}$. Then use your work in parts 4a and 4b to justify the picture in a paragraph consisting of complete sentences.

5. (20 points) Let L be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula

$$Ref_L(\vec{x}) \quad = \quad \frac{2(\vec{x} \cdot \vec{v})}{\vec{v} \cdot \vec{v}} \vec{v} - \vec{x}. \tag{1}$$

(a) Use the formula (1) to find the standard matrix A of Ref_L.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane about the origin $\frac{\pi}{3}$ radians (i.e., 60 degrees) counter-clockwise. Find the standard matrix B of the rotation T. Hint: $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.

(c) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $S(\vec{x}) = Ref_L(T(\vec{x}))$ (i.e., rotation followed by reflection). Express the standard matrix C of S in terms of the matrices A of Ref_L and B of T.
$C =$________.

(d) Use the expression in part 5c to compute the matrix C. Note: The answer is $C = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

(e) Let $\widetilde{L}$ be the line through the origin and the vector $\vec{w} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. The matrix C in part 5d is the matrix of the reflection $Ref_{\widetilde{L}}$ with respect to this new line $\widetilde{L}$. You need **not** prove this fact. Use this fact and your work above in order to express the rotation T in terms of the reflections Ref_L and $Ref_{\widetilde{L}}$.

$T(\vec{x}) =$____________________________. Justify your answer!

6. (15 points)

(a) Is the vector $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$? Justify your answer!

(b) Let A be a 4×3 matrix such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ has a unique solution.

i. What is the rank of A? Justify your answer!

ii. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Is the image of T equals the whole of $\mathbb{R}^4$? Justify your answer!