Practice Problems: Solutions and hints

1. (8 points) Which of the following subsets $S \subseteq V$ are subspaces of $V$? Write YES if $S$ is a subspace and NO if $S$ is not a subspace.
   
a. (2 pts) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \leq y \leq z \right\}$

   $NO$: $S$ is not closed under scalar multiplication. For example, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in S$, but $-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \notin S$.

   b. (2 pts) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$

   $YES$: $S = \text{ker} \left[ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right]$.

   c. (2 pts) $S$ is the set of vectors of the form $\begin{pmatrix} a + 2b + 3c \\ c \\ 0 \end{pmatrix}$.

   $YES$: $S = \text{im} \left[ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right]$.

   d. (2 pts) $S$ is the set polynomials $p$ in $\mathcal{P}_3$ such that $p'(2) = 0$.

   $YES$: $S = \ker(T)$, where $T : \mathcal{P}_3 \to \mathcal{P}_3$ is the linear transformation $T(p) = p'(2)$.

2. (10 points) Solve the following system of linear equations.

   $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

   Use Gaussian elimination. The solutions are $\vec{x} = \begin{pmatrix} 1 - t \\ -1 - 2t \\ t \end{pmatrix}$ for $t \in \mathbb{R}$.

3. (10 points) Solve the following system of linear equations.

   $x - z = 1$
   $x + 2y + 3z = 11$.

   Use Gaussian elimination. The solutions are $x = 1 + t$, $y = 5 - 2t$, $z = t$ for $t \in \mathbb{R}$.

4. (7 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation counterclockwise about the origin in $\mathbb{R}^2$ by $\frac{\pi}{4}$ radians or $45^\circ$.
   
a. (3 pts) Compute the matrix that represents $T$. 

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The matrix that represents a counterclockwise rotation in \( \mathbb{R}^2 \) by angle \( \theta \) is given by
\[
A = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}.
\]
It follows that the matrix that represents \( T \) is
\[
\begin{bmatrix}
\cos(\pi/4) & -\sin(\pi/4) \\
\sin(\pi/4) & \cos(\pi/4)
\end{bmatrix}
= \begin{bmatrix}
1/\sqrt{2} & -1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}.
\]

b. (2 pts) Is \( T \) an isomorphism?
Yes. An isomorphism is an invertible linear transformation. It is clear that counterclockwise rotation by \( \pi/4 \) is an invertible transformation (the inverse is clockwise rotation by \( \pi/4 \)).

c. (2 pts) Is \( T \) diagonalizable?
No. Either argue geometrically that \( T \) has no eigenvectors, or show that \( T \) has no eigenvalues since the characteristic polynomial, \( \lambda^2 - \sqrt{2}\lambda + 1 \) has no real roots.

5. (6 points) Let \( A \) be a \( n \times n \) orthogonal matrix.
a. (2 pts) What is the rank of \( A \)?
Since \( A \) preserves lengths, \( \ker(A) = \{0\} \). Thus Rank-Nullity theorem implies that \( \text{rank}(A) = n \).

b. (2 pts) What are the possible values for \( \det(A) \)?
Since \( A \) is orthogonal, \( A^T A = I \). It follows that \( \det(A^T) \det(A) = (\det(A))^2 = 1 \). Hence \( \det(A) = \pm 1 \).

c. (2 pts) If \( \lambda \) is an eigenvalue for \( A \), what are the possible values for \( \lambda \)?
Since \( A \) preserves lengths, if \( \vec{v} \) is an eigenvector with associated eigenvalue \( \lambda \), then \( \|A\vec{v}\| = ||\lambda\vec{v}|| = |\lambda||\vec{v}|| = \|\vec{v}\| \). It follows that \( \lambda = \pm 1 \).

6. (13 points) Consider the linear transformation \( T : \mathcal{P}_2 \to \mathcal{P}_2 \) given by \( (T(f))(x) = f(2x - 1) \).
Let \( B \) be the ordered basis \( B = (1, x, x^2) \).
a. (3 pts) Compute \( \text{Mat}^B_B(T) \).
\[
A = \text{Mat}^B_B(T) = \begin{bmatrix}
[T(1)]_B & [T(x)]_B & [T(x^2)]_B \\
[1]_B & [2x - 1]_B & [(2x - 1)^2]_B \\
0 & 2 & -4 \\
0 & 0 & 4
\end{bmatrix}.
\]

b. (3 pts) Compute the eigenvalues of \( T \).
The eigenvalues are the roots of the characteristic polynomial \( f_A(\lambda) = (1 - \lambda)(2 - \lambda)(4 - \lambda) \). Hence the eigenvalues are 1, 2, and 4.
c. (3 pts) Is $T$ diagonalizable?

Yes. $T$ is diagonalizable because it has three distinct eigenvalues.

d. (4 pts) Compute the eigenspaces of $T$. Make sure your answers are expressed as subspaces of $P_2$.

Compute $E_{\lambda}$ as $\ker(A - \lambda I)$. Then convert each $E_{\lambda}$ to a subspace of $P_2$. You should get $E_1 = \text{span}(1)$, $E_2 = \text{span}(x - 1)$, and $E_4 = \text{span}(x^2 - 2x + 1)$.

7. (12 points) Two interacting populations of foxes and hares can be modeled by the equations

$$h(t + 1) = 4h(t) - 2f(t)$$
$$f(t + 1) = h(t) + f(t).$$

a. (4 pts) Find a matrix $A$ such that

$$
\begin{pmatrix}
  h(t + 1) \\
  f(t + 1)
\end{pmatrix} = A
\begin{pmatrix}
  h(t) \\
  f(t)
\end{pmatrix}.
\]

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

b. (8 pts) Find a formula for $h(t)$ and $f(t)$.

If we let $\vec{x}(t) = \begin{pmatrix} h(t) \\ f(t) \end{pmatrix}$, then $\vec{x}(t) = A^t\vec{x}(0)$. To find closed formulas for $h(t)$ and $f(t)$ we must first diagonalize $A$. We compute that

$$f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Thus the eigenvalues are 2 and 3. We must find the associated eigenvectors.

$$E_2 = \ker(A - 2I) = \ker \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

$= \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, and

$$E_3 = \ker(A - 3I) = \ker \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$$

$= \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. 

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It follows that $\vec{x}(t) = SD^tS^{-1}\vec{x}(0)$, where $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Thus

\[
\begin{pmatrix} h(t) \\ f(t) \end{pmatrix} = -\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^t \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{pmatrix} h_0 \\ f_0 \end{pmatrix}
\]

where $h_0 = h(0)$ and $f_0 = f(0)$,

\[
= -\begin{bmatrix} 2^t \\ 2^t \end{bmatrix} \begin{bmatrix} 2^t(h_0 - 2f_0) \\ 2^t(h_0 - 2f_0) + 3^t(-h_0 + f_0) \end{bmatrix}
\]

\[
= -\begin{bmatrix} 2^t(h_0 - 2f_0) - 2(3^t(-h_0 + f_0)) \\ 2^t(h_0 - 2f_0) - 3^t(-h_0 + f_0) \end{bmatrix}.
\]

$h(t) = -2^t(h_0 - 2f_0) - 2(3^t(-h_0 + f_0)$ and $f(t) = -2^t(h_0 - 2f_0) - 3^t(-h_0 + f_0)$

8. (10 points) Let $A$ be a $3 \times 3$ matrix such that

\[
A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]

has $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as solutions. Find another solution. Explain.

It follows that $A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = A \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ is also a solution to $A\vec{x}$ for every $c \in \mathbb{R}$. For example, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ is a solution.

9. (12 points) Let $T : \mathbb{R}^{9 \times 10} \rightarrow \mathbb{R}^9$ be the map defined by $T(A) = Ae_1$.

a. (4 pts) Show that $T$ is a linear transformation.

We must verify 3 things:

1. $T(Z) = \vec{0}$, where $Z$ is the $9 \times 10$ zero matrix.

   This is clear.

2. $T(A + B) = T(A) + T(B)$

   This follows from $T(A + B) = (A + B)e_1 = Ae_1 + Be_1 = T(A) + T(B)$.

3. $T(kA) = kT(A)$

   This follows from $T(kA) = (kA)e_1 = k(Ae_1) = kT(A)$.
b. (4 pts) What is the rank of $T$?

The rank can be interpreted as the dimension of the image of $T$. It is clear that the image of $T$ is all of $\mathbb{R}^9$. Thus the rank is $9$.

c. (4 pts) State the Rank-Nullity Theorem and use it to compute the nullity of $T$.

The Rank-Nullity theorem states that: Given a linear transformation $T : V \to W$,

$$\text{rank}(T) + \text{null}(T) = \dim(V).$$

Hence, $\text{null}(T) = \dim(V) - \text{rank}(T) = 90 - 9 = 89$.

10. (12 points)
   a. (4 pts) Give the definition of the phrase $V$ is a subspace of $\mathbb{R}^n$.

$V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. $\vec{0} \in V$.

2. if $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} \in V$.

3. if $\vec{v} \in V$ and $k \in \mathbb{R}$, then $k\vec{v} \in V$.

b. (8 pts) Let $V$ be a subspace of $\mathbb{R}^n$. Prove that $V^\perp = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V\}$ is a subspace of $\mathbb{R}^n$.

We just have to show that $V^\perp$ satisfies the conditions above.

1. $\vec{0} \in V^\perp$.

   $\vec{0} \cdot \vec{u} = 0$ for every $\vec{u} \in V$.

2. if $\vec{v}, \vec{w} \in V^\perp$, then $\vec{v} + \vec{w} \in V^\perp$.

   If $\vec{v}, \vec{w} \in V^\perp$, then $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = 0 + 0 = 0$ for every $\vec{u} \in V$.

3. if $\vec{v} \in V$ and $k \in \mathbb{R}$, then $k\vec{v} \in V^\perp$.

   If $\vec{v} \in V$ and $k \in \mathbb{R}$, then $(k\vec{v}) \cdot \vec{u} = k(\vec{v} \cdot \vec{u}) = 0$ for every $\vec{u} \in V$. 