Practice Problems: Solutions and hints

1. (*8 points*) Which of the following subsets $S \subseteq V$ are subspaces of V? Write *YES* if S is a subspace and *NO* if S is not a subspace.

a. (*2 pts*) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \le y \le z \right\}$

NO: S is not closed under scalar multiplication. For example, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in S$, but $-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \notin S$.

b. (*2 pts*) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$

YES: $S = \ker\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right)$.

c. (*2 pts*) S is the set of vectors of the form $\begin{pmatrix} a + 2b + 3c \\ c \\ 0 \end{pmatrix}$.

YES: $S = \operatorname{im}\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$.

d. (*2 pts*) S is the set polynomials p in $\mathcal{P}_3$ such that $p'(2) = 0$.

YES: $S = \ker(T)$, where $T : \mathcal{P}_3 \to \mathcal{P}_3$ is the linear transformation $T(p) = p'(2)$.

2. (*10 points*) Solve the following system of linear equations.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Use Gaussian elimination. The solutions are $\vec{x} = \begin{pmatrix} 1 - t \\ -1 - 2t \\ t \end{pmatrix}$ for $t \in \mathbb{R}$.

3. (*10 points*) Solve the following system of linear equations.

$$\begin{aligned} x - z &= 1 \\ x + 2y + 3z &= 11. \end{aligned}$$

Use Gaussian elimination. The solutions are $x = 1 + t$, $y = 5 - 2t$, $z = t$ for $t \in \mathbb{R}$.

4. (*7 points*) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation counterclockwise about the origin in $\mathbb{R}^2$ by $\frac{\pi}{4}$ radians or $45°$.

a. (*3 pts*) Compute the matrix that represents T.

The matrix that represents a counterclockwise rotation in $\mathbb{R}^2$ by angle θ is given by

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

It follows that the matrix that represents T is

$$\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/sqrt2 \end{bmatrix}.$$

b. (*2 pts*) Is T an isomorphism?

Yes. An isomorphism is an invertible linear transformation. It is clear that counterclockwise rotation by $\pi/4$ is an invertible transformation (the inverse is clockwise rotation by $\pi/4$).

c. (*2 pts*) Is T diagonalizable?

No. Either argue geometrically that T has no eigenvectors, or show that T has no eigenvalues since the characteristic polynomial, $\lambda^2 - \sqrt{2}\lambda + 1$ has no real roots.

5. (*6 points*) Let A be a $n \times n$ orthogonal matrix.

a. (*2 pts*) What is the rank of A?

Since A preserves lengths, $\ker(A) = \{0\}$. Thus Rank-Nullity theorem implies that $\operatorname{rank}(A) = n$.

b. (*2 pts*) What are the possible values for $\det(A)$?

Since A is orthogonal, $A^T A = I$. It follows that $\det(A^T)\det(A) = (\det(A))^2 = 1$. Hence $\det(A) = \pm 1$.

c. (*2 pts*) If λ is an eigenvalue for A, what are the possible values for λ?

Since A preserves lengths, if $\vec{v}$ is an eigenvector with associated eigenvalue λ, then $\|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\| = \|\vec{v}\|$. It follows that $\lambda = \pm 1$.

6. (*13 points*) Consider the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $(T(f))(x) = f(2x-1)$. Let $\mathcal{B}$ be the ordered basis $\mathcal{B} = (1, x, x^2)$.

a. (*3 pts*) Compute $\operatorname{Mat}_{\mathcal{B}}^{\mathcal{B}}(T)$.

$$\begin{aligned} A = \operatorname{Mat}_{\mathcal{B}}^{\mathcal{B}}(T) &= \begin{bmatrix} [T(1)]_{\mathcal{B}} & [T(x)]_{\mathcal{B}} & [T(x^2)]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} [1]_{\mathcal{B}} & [2x-1]_{\mathcal{B}} & [(2x-1)^2]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}. \end{aligned}$$

b. (*3 pts*) Compute the eigenvalues of T.

The eigenvalues are the roots of the characteristic polynomial $f_A(\lambda) = (1-\lambda)(2-\lambda)(4-\lambda)$. Hence the eigenvalues are 1, 2,and 4

c. (*3 pts*) Is T diagonalizable?

Yes. T is diagonalizable because it has three distinct eigenvalues.

d. (*4 pts*) Compute the eigenspaces of T. Make sure your answers are expressed as subspaces of $\mathcal{P}_2$.

Compute E_λ as $\ker(A - \lambda I)$. Then convert each E_λ to a subspace of $\mathcal{P}_2$. You should get $E_1 = \operatorname{span}(1)$, $E_2 = \operatorname{span}(x-1)$, and $E_4 = \operatorname{span}(x^2 - 2x + 1)$.

7. (*12 points*) Two interacting populations of foxes and hares can be modeled by the equations

$$h(t+1) = 4h(t) - 2f(t)$$
$$f(t+1) = h(t) + f(t).$$

a. (*4 pts*) Find a matrix A such that

$$\begin{pmatrix} h(t+1) \\ f(t+1) \end{pmatrix} = A \begin{pmatrix} h(t) \\ f(t) \end{pmatrix}.$$

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

b. (*8 pts*) Find a formula for $h(t)$ and $f(t)$.

If we let $\vec{x}(t) = \begin{pmatrix} h(t) \\ f(t) \end{pmatrix}$, then $\vec{x}(t) = A^t \vec{x}(0)$. To find closed formulas for $h(t)$ and $f(t)$ we must first diagonalize A. We compute that

$$f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Thus the eigenvalues are 2 and 3. We must find the associated eigenvectors.

$$\begin{aligned} E_2 = \ker(A - 2I) &= \ker\left(\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\right) \\ &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad \text{and} \\ E_3 = \ker(A - 3I) &= \ker\left(\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right) \\ &= \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}. \end{aligned}$$

It follows that $\vec{x}(t) = SD^tS^{-1}\vec{x}(0)$, where $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Thus

$$\begin{aligned}
\begin{pmatrix} h(t) \\ f(t) \end{pmatrix} &= -\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} h_0 \\ f_0 \end{pmatrix} \quad \text{where } h_0 = h(0) \text{ and } f_0 = f(0), \\
&= -\begin{bmatrix} 2^t & 2(3^t) \\ 2^t & 3^t \end{bmatrix}\begin{pmatrix} h_0 - 2f_0 \\ -h_0 + f_0 \end{pmatrix} \\
&= -\begin{pmatrix} 2^t(h_0 - 2f_0) + 2(3^t)(-h_0 + f_0) \\ 2^t(h_0 - 2f_0) + 3^t(-h_0 + f_0) \end{pmatrix} \\
&= \begin{pmatrix} -2^t(h_0 - 2f_0) - 2(3^t)(-h_0 + f_0) \\ -2^t(h_0 - 2f_0) - 3^t(-h_0 + f_0) \end{pmatrix}.
\end{aligned}$$

$$h(t) = -2^t(h_0 - 2f_0) - 2(3^t)(-h_0 + f_0) \quad \text{and} \quad f(t) = -2^t(h_0 - 2f_0) - 3^t(-h_0 + f_0)$$

8. (*10 points*) Let A be a 3×3 matrix such that

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ as solutions. Find another solution. Explain.

It follows that $A\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right] = A\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ is also a solution to $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for every $c \in \mathbb{R}$. For example, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ is a solution.

9. (*12 points*) Let $T : \mathbb{R}^{9\times 10} \to \mathbb{R}^9$ be the map defined by

$$T(A) = A\vec{e}_1.$$

a. (*4 pts*) Show that T is a linear transformation.

We must verify 3 things:

1. $T(Z) = \vec{0}$, where Z is the 9×10 zero matrix.
   This is clear.
2. $T(A + B) = T(A) + T(B)$
   This follows from $T(A + B) = (A + B)\vec{e}_1 = A\vec{e}_1 + Be_1 = T(A) + T(B)$.
3. $T(kA) = kT(A)$
   This follows from $T(kA) = (kA)\vec{e}_1 = k(A\vec{e}_1) = kT(A)$.

b. (*4 pts*) What is the rank of T?

The rank can be interpreted as the dimension of the image of T. It is clear that the image of T is all of $\mathbb{R}^9$. Thus the rank if 9. **c.** (*4 pts*) State the Rank-Nullity Theorem and use it to compute the nullity of T.

The Rank-Nullity theorem states that: Given a linear transformation $T : V \to W$,

$$\text{rank}(T) + \text{null}(T) = \dim(V).$$

Hence, $\text{null}(T) = \dim(V) - \text{rank}(T) = 90 - 9 = 89$.

10. (*12 points*)

a. (*4 pts*) Give the definition of the phrase V *is a subspace of* $\mathbb{R}^n$.

$V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. $\vec{0} \in V$.
2. if $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} \in V$.
3. if $\vec{v} \in V$ and $k \in \mathbb{R}$, then $k\vec{v} \in V$.

b. (*8 pts*) Let V be a subspace of $\mathbb{R}^n$. Prove that $V^\perp = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \cdot \vec{v} = 0 \quad \text{for every } \vec{v} \in V\}$ is a subspace of $\mathbb{R}^n$.

We just have to show that $V^\perp$ satisfies the conditions above.

1. $\vec{0} \in V^\perp$.

   $\vec{0} \cdot \vec{u} = 0$ for every $\vec{u} \in V$.

2. if $\vec{v}, \vec{w} \in V^\perp$, then $\vec{v} + \vec{w} \in V^\perp$.

   If $\vec{v}, \vec{w} \in V^\perp$, then $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = 0 + 0 = 0$ for every $\vec{u} \in V$.

3. if $\vec{v} \in V$ and $k \in \mathbb{R}$, then $k\vec{v} \in V^\perp$.

   If $\vec{v} \in V$ and $k \in \mathbb{R}$, then $(k\vec{v}) \cdot \vec{u} = k(\vec{v} \cdot \vec{u}) = 0$ for every $\vec{u} \in V$.