Math 235 Solution Midterm 2 Spring 2008

1. (20 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 2 & 2 & 2 \\ 2 & 1 & 4 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Determine the rank of A, $\dim(\ker(A))$, and $\dim(\mathrm{im}(A))$. Explain how these are determined by the matrix B.

Answer: $\mathrm{rank}(A) =$ number of pivots in $B = 3$.
$\dim(\ker(A)) =$ number of free variable $= 6 - 3 = 3$.
$\dim(im(A)) = \mathrm{rank}(A) = 3$.

b) Find a basis for the kernel $\ker(A)$ of A.

Answer: The variables x_3, x_4, and x_5 are free. Expressing the basic variables in terms of the free variables, we get that the general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ -2x_3 - x_4 - x_5 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$$

$x_3\vec{v}_1 + x_4\vec{v}_2 + x_5\vec{v}_3$. The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are clearly linearly independent, and so a basis of $\ker(A)$.

c) Find a basis for the image $\mathrm{im}(A)$ of A.

Answer: The pivot columns of A are the first, second, and sixth, so

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \ a_6 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$ are a basis for $im(A)$.

d) Does the vector $b := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ belong to the image of A? Use part c to minimize your computations. **Justify** your answer!

Answer: The vector b is a linear combination of the basis elements a_1, a_2, a_6 of $im(A)$, if and only if the vector equation $x_1a_1 + x_2a_2 + x_3a_6 = b$ is consistent. Row reduce the augmented matrix:

$$(a_1 a_2 a_6 \mid b) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ We get a pivot in the rightmost column, so the equation is inconsistent. Hence, b does not belong to $im(A)$.

2. (12 points) Let A be a 4×5 matrix with columns $\vec{a}_1, \ldots, \vec{a}_5$. We are given that the vector

$\vec{x} := \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 5 \end{pmatrix}$ belongs to the kernel of A and the vectors $v_1 := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $v_2 := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ span the image of A.

a) Express $\vec{a}_5$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$.

Answer: $0 = A\vec{x} = 3\vec{a}_1 + 2\vec{a}_2 + \vec{a}_3 + 4\vec{a}_4 + 5\vec{a}_5$. Hence, $a_5 = \frac{-3}{5}\vec{a}_1 - \frac{2}{5}\vec{a}_2 - \frac{1}{5}\vec{a}_3 - \frac{4}{5}\vec{a}_4$.

b) Determine $\dim(\mathrm{im}(A))$. Justify your answer.

Answer: The vectors v_1 and v_2 are linearly independent, since neither one is a scalar multiple of the other, and they span $im(A)$, by assumption, hence they constitute a basis of $im(A)$, consisting of two elements. Thus, $\dim(im(A)) = 2$.

c) Determine $\dim(\ker(A))$. Justify your answer.

Answer: The Rank-Nullity Theorem asserts that $\dim(\ker(A)) + \dim(im(A)) = 5$. Hence, $\dim(\ker(A)) = 5 - \dim(im(A)) = 5 - 2 = 3$.

3. (20 points) Let $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\beta := \{v_1, v_2\}$ the basis of $\mathbb{R}^2$.

(a) Find a vector w in $\mathbb{R}^2$, such that the coordinate vector of w with respect to the basis β is $[w]_\beta = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. **Answer:** $w = 2v_1 + 3v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(b) Let $w_1 := \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $w_2 := \begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Find the coordinate vectors $[w_1]_\beta$ and $[w_2]_\beta$ with respect to the basis β.

Answer: $w_1 = 2v_1 + 0v_2$, so $[w_1]_\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$w_2 = x_1 v_1 + x_2 v_2$, and we find the coefficients x_i by row reduction:
$(v_1 v_2 \mid w_2) = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}$. So $[w_2]_\beta = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(c) Let $A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation given by $T(\vec{x}) = A\vec{x}$. Note that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Use this information and your work in part 3b to find the matrix B of T with respect to the basis β of $\mathbb{R}^2$.

Answer: $B = ([T(v_1)]_\beta [T(v_2)]_\beta) = ([w_1]_\beta [w_2]_\beta) = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$.

(d) Let $\tilde{v}_1$, $\tilde{v}_2$, be two linearly independent vectors in $\mathbb{R}^2$, and $\widetilde{S} := (\tilde{v}_1 \tilde{v}_2)$ the 2×2 matrix with $\tilde{v}_j$ as its j-th column. Let $\widetilde{B}$ be the matrix of the linear transformation T in part 3c, with respect to the new basis $\tilde{\beta} := \{\tilde{v}_1, \tilde{v}_2\}$. Express $\widetilde{B}$ in terms of the matrices A and $\widetilde{S}$. **Answer:** $\widetilde{B} = \widetilde{S}^{-1} A \widetilde{S}$

(e) Let $S := (v_1 v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Express $\widetilde{B}$ in terms of the matrices S, $\widetilde{S}$, and B. Your final answer should not involve the matrix A. Hint: Express first A in terms of S and B. Then express A in terms of $\widetilde{S}$ and $\widetilde{B}$.

Answer: $A = SBS^{-1}$. Substituting the right hand side for A in the answer to part 3d, we get $\widetilde{B} = \widetilde{S}^{-1} SBS^{-1} \widetilde{S}$. The above equality shows that B and $\widetilde{B}$ are similar, since $\widetilde{S}^{-1} S$ is the inverse of $S^{-1}\widetilde{S}$.

4. (18 points) Denote the vector space of 2×2 matrices by $R^{2\times 2}$. Let $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ the linear transformation given by $T(M) = AM - MA$.

a) Find the matrix B of T in the basis
$\beta := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $R^{2\times 2}$.

Answer:

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -b & 0 \\ a-d & b \end{pmatrix}.$$

$$B = \left(\left[T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\right]_\beta \left[T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right]_\beta \left[T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\right]_\beta \left[T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\right]_\beta \right)$$

$$= \left(\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_\beta \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]_\beta \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]_\beta \left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

b) Find a basis for $\ker(B)$. **Answer:** Row reducing B we get the basis: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

c) Find a basis for $\ker(T)$.

Answer: We simply write the elements of $R^{2\times 2}$, whose coordinate vectors we found in part b. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

d) Find a basis for $\operatorname{im}(B)$.

Answer: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

e) Find a basis for $\operatorname{im}(T)$.

Answer: We simply write the elements of $R^{2\times 2}$, whose coordinate vectors we found in part d. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

5. (10 points) Let V and W be two vector spaces and $T : V \to W$ a linear transformation from V to W. Let p be a positive integer and $\{f_1, \ldots, f_p\}$ a linearly **dependent** subset of V consisting of p elements. Show the the subset $\{T(f_1), \ldots, T(f_p)\}$ of W is linearly dependent as well. Note: Provide an argument that works for general vector spaces, starting with the definition of linear dependence.

Answer: The set $\{f_1, \ldots, f_p\}$ is linearly dependent, if the equation $0 = c_1 f_1 + \cdots + c_p f_p$, with the scalar coefficients c_i as unknowns, has a solution with at least one non-zero c_i. Choose such a solution and apply T to both sides of the equation to get:

$$0 = T(0) = T(c_1 f_1 + \cdots + c_p f_p) = c_1 T(f_1) + \cdots + c_p T(f_p),$$

where in the first and last equalities we used the linearity properties of T. We conclude that the equation $0 = c_1 T(f_1) + \cdots + c_p T(f_p)$ has a solution with at least one non-zero c_i. Hence, the subset $\{T(f_1), \ldots, T(f_p)\}$ of W is linearly dependent.

6. (20 points) Let $C^\infty(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, having derivatives of all orders. Denote by V the subspace of $C^\infty(\mathbb{R})$ spanned by the functions $f_1(x) = e^x$, $f_2(x) = e^{2x}$, and $f_3(x) = e^{3x}$. Let $T : V \to \mathbb{R}^3$ be the transformation given by $T(f) := \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix}$.

   (a) Show that the transformation T is linear. In other words, verify the following identities, for any two elements f, g of V, and for every scalar k.

      i. $T(f+g) = T(f) + T(g)$. **Answer:** $T(f+g) = \begin{pmatrix} (f+g)(0) \\ (f+g)'(0) \\ (f+g)''(0) \end{pmatrix} = \begin{pmatrix} f(0)+g(0) \\ f'(0)+g'(0) \\ f''(0)+g''(0) \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} + \begin{pmatrix} g(0) \\ g'(0) \\ g''(0) \end{pmatrix} = T(f) + T(g)$

      ii. $T(kf) = kT(f)$. **Answer:** $T(kf) = \begin{pmatrix} kf(0) \\ kf'(0) \\ kf''(0) \end{pmatrix} = k\begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} = kT(f)$.

   (b) Show that the subset $\{T(f_1), T(f_2), T(f_3)\}$ of $\mathbb{R}^3$ is linearly independent. Hint: Recall that the chain rule yields $(e^{2x})' = 2e^{2x}$, $(e^{2x})'' = 2^2e^{2x}$, and so $f_2''(0) = 4$.

   Answer: $T(f_1) = \begin{pmatrix} e^0 \\ e^0 \\ e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $T(f_2) = \begin{pmatrix} e^0 \\ 2e^0 \\ 4e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$,

   $T(f_3) = \begin{pmatrix} e^0 \\ 3e^0 \\ 9e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$. Row reducing, we get:

   $(T(f_1)T(f_2)T(f_3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

   We get a pivot in every column, so the columns $T(f_1), T(f_2), T(f_3)$ are linearly independent, and a pivot in every row, so $T(f_1), T(f_2), T(f_3)$ span the whole of $\mathbb{R}^3$.

   (c) Show that $\mathrm{im}(T)$ is the whole of $\mathbb{R}^3$.

   Answer: $im(T) = \mathrm{span}\{T(f_1), T(f_2), T(f_3)\}$, and the latter was shown to be the whole of $\mathbb{R}^3$ in the previous part.

   (d) Show the the subset $\{e^x, e^{2x}, e^{3x}\}$ of V is linearly independent. Hint: Use part 6b and question 5.

   Answer: We argue as in question 5. Suppose $c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$. Applying T to both sides we get

$$c_1T(e^x) + c_2T(e^{2x}) + c_3T(e^{3x}) = \vec{0}.$$

   The vectors $T(e^x), T(e^{2x}), T(e^{3x})$ in $\mathbb{R}^3$ are linearly independent, by part 6b. Hence, $c_1 = c_2 = c_3 = 0$. Hence, the subset $\{e^x, e^{2x}, e^{3x}\}$ of V is linearly independent.

   (e) Show that $T : V \to \mathbb{R}^3$ is an isomorphism.

   Answer: It suffices to show that $\ker(T) = \{0\}$ and $im(T) = \mathbb{R}^3$. The equality $im(T) = \mathbb{R}^3$ was shown in part 6c. The Rank-Nullity-Theorem yields the equality $\dim(\ker(T)) + \dim(im(T)) = \dim(V)$. The set $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent, by part 6d, and spans V, by definition of V, and is thus a basis for V. The vector space V is three-dimensional, having a basis consisting of three elements. Hence, $\dim(\ker(T)) = \dim(V) - \dim(im(T)) = 3 - 3 = 0$. Thus, $\ker(T) = \{0\}$.