1. (20 points) You are given below the matrix \( A \) together with its row reduced echelon form \( B \)

\[
A = \begin{pmatrix}
1 & 1 & 3 & 0 & 1 & 0 \\
0 & 2 & 4 & 2 & 2 & 2 \\
2 & 1 & 4 & -1 & 1 & 0 \\
1 & 1 & 3 & 0 & 1 & 1 \\
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

a) Determine the rank of \( A \), \( \dim(\ker(A)) \), and \( \dim(\text{im}(A)) \). Explain how these are determined by the matrix \( B \).

**Answer:**

- \( \text{rank}(A) = \) number of pivots in \( B \) = 3.
- \( \dim(\ker(A)) = \) number of free variable = 6 − 3 = 3.
- \( \dim(\text{im}(A)) = \text{rank}(A) = 3 \).

b) Find a basis for the kernel \( \ker(A) \) of \( A \).

**Answer:**

The variables \( x_3, x_4, \) and \( x_5 \) are free. Expressing the basic variables in terms of the free variables, we get that the general solution is:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix} = x_3 \begin{pmatrix}
-1 \\
-2 \\
1 \\
0 \\
0 \\
0
\end{pmatrix} + x_4 \begin{pmatrix}
1 \\
1 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} + x_5 \begin{pmatrix}
1 \\
-1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} = x_3 \vec{v}_1 + x_4 \vec{v}_2 + x_5 \vec{v}_3.
\]

The vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are clearly linearly independent, and so a basis of \( \ker(A) \).

c) Find a basis for the image \( \text{im}(A) \) of \( A \).

**Answer:**

The pivot columns of \( A \) are the first, second, and sixth, so

\[
a_1 = \begin{pmatrix}
1 \\
0 \\
2 \\
1
\end{pmatrix}, \quad a_2 = \begin{pmatrix}
1 \\
2 \\
1
\end{pmatrix}, \quad a_6 = \begin{pmatrix}
0 \\
2 \\
0 \\
1
\end{pmatrix}
\]

are a basis for \( \text{im}(A) \).

d) Does the vector \( b := \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} \) belong to the image of \( A \)? Use part c to minimize your computations. **Justify** your answer!

**Answer:**

The vector \( b \) is a linear combination of the basis elements \( a_1, a_2, a_6 \) of \( \text{im}(A) \), if and only if the vector equation \( x_1 a_1 + x_2 a_2 + x_3 a_6 = b \) is consistent. Row reduce the augmented matrix:

\[
(a_1 a_2 a_6 \mid b) = \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 2 & 2 & 0 \\
2 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix} \sim \cdots \sim \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We get a pivot in the rightmost column, so the equation is inconsistent. Hence, \( b \) does not belong to \( \text{im}(A) \).

2. (12 points) Let \( A \) be a \( 4 \times 5 \) matrix with columns \( \vec{a}_1, \ldots, \vec{a}_5 \). We are given that the vector
3. (20 points) Let \( x := \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 5 \end{pmatrix} \) belongs to the kernel of \( A \) and the vectors \( v_1 := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \) and \( v_2 := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) span the image of \( A \).

a) Express \( a_5 \) as a linear combination of \( a_1, a_2, a_3, a_4 \).

**Answer:** \( 0 = Ax = 3a_1 + 2a_2 + a_3 + 4a_4 + 5a_5 \). Hence, \( a_5 = -\frac{3}{5} a_1 - \frac{2}{5} a_2 - \frac{1}{5} a_3 - \frac{4}{5} a_4 \).

b) Determine \( \dim(\text{im}(A)) \). Justify your answer.

**Answer:** The vectors \( v_1 \) and \( v_2 \) are linearly independent, since neither one is a scalar multiple of the other, and they span \( \text{im}(A) \), by assumption, hence they constitute a basis of \( \text{im}(A) \), consisting of two elements. Thus, \( \dim(\text{im}(A)) = 2 \).

c) Determine \( \dim(\ker(A)) \). Justify your answer.

**Answer:** The Rank-Nullity Theorem asserts that \( \dim(\ker(A)) + \dim(\text{im}(A)) = 5 \). Hence, \( \dim(\ker(A)) = 5 - \dim(\text{im}(A)) = 5 - 2 = 3 \).

3. (20 points) Let \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \) and \( \beta := \{v_1, v_2\} \) the basis of \( \mathbb{R}^2 \).

a) Find a vector \( w \) in \( \mathbb{R}^2 \), such that the coordinate vector of \( w \) with respect to the basis \( \beta \) is \([w]_\beta = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \).

**Answer:** \( w = 2v_1 + 3v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \).

b) Let \( w_1 := \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) and \( w_2 := \begin{pmatrix} -3 \\ -4 \end{pmatrix} \). Find the coordinate vectors \([w_1]_\beta\) and \([w_2]_\beta\) with respect to the basis \( \beta \).

**Answer:** \( w_1 = 2v_1 + 0v_2 \), so \([w_1]_\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \).

\( w_2 = x_1 v_1 + x_2 v_2 \), and we find the coefficients \( x_i \) by row reduction:

\[
\begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}.
\]

So \([w_2]_\beta = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \).

d) Let \( A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \) and \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) the linear transformation given by \( T(x) = Ax \). Note that \( w_1 = T(v_1) \) and \( w_2 = T(v_2) \). Use this information and your work in part 3b to find the matrix \( B \) of \( T \) with respect to the basis \( \beta \) of \( \mathbb{R}^2 \).

**Answer:** \( B = ([T(v_1)]_\beta[T(v_2)]_\beta) = ([w_1]_\beta[w_2]_\beta) = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} \).

(e) Let \( \tilde{v}_1, \tilde{v}_2 \), be two linearly independent vectors in \( \mathbb{R}^2 \), and \( \tilde{S} := (\tilde{v}_1 \tilde{v}_2) \) the \( 2 \times 2 \) matrix with \( \tilde{v}_j \) as its \( j \)-th column. Let \( \tilde{B} \) be the matrix of the linear transformation \( T \) in part 3c, with respect to the new basis \( \beta := \{\tilde{v}_1, \tilde{v}_2\} \). Express \( \tilde{B} \) in terms of the matrices \( A \) and \( \tilde{S} \).

**Answer:** \( \tilde{B} = \tilde{S}^{-1}AS \).

e) Let \( S := (v_1v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). Express \( \tilde{B} \) in terms of the matrices \( S, \tilde{S}, \) and \( B \).

**Answer:** \( A = SBS^{-1} \). Substituting the right hand side for \( A \) in the answer to part 3d, we get \( \tilde{B} = \tilde{S}^{-1}SBS^{-1}\tilde{S} \). The above equality shows that \( B \) and \( \tilde{B} \) are similar, since \( S^{-1}S \) is the inverse of \( S^{-1} \tilde{S} \).
4. (18 points) Denote the vector space of $2 \times 2$ matrices by $R^{2 \times 2}$. Let $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T : R^{2 \times 2} \to R^{2 \times 2}$ the linear transformation given by $T(M) = AM - MA$.

a) Find the matrix $B$ of $T$ in the basis 
$\beta := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $R^{2 \times 2}$.

**Answer:** 
$T\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -b & 0 \\ a-d & b \end{pmatrix}$.

$B = \left[ \begin{pmatrix} T\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right]_\beta \begin{pmatrix} T\left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \right]_\beta \begin{pmatrix} T\left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \right]_\beta \begin{pmatrix} T\left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \right]_\beta \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}$.

b) Find a basis for $\ker(B)$. **Answer:** Row reducing $B$ we get the basis: $\left( \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$.

c) Find a basis for $\ker(T)$.

**Answer:** We simply write the elements of $R^{2 \times 2}$, whose coordinate vectors we found in part b. $\left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$.

d) Find a basis for $\text{im}(B)$.

**Answer:** $\left( \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$.

e) Find a basis for $\text{im}(T)$.

**Answer:** We simply write the elements of $R^{2 \times 2}$, whose coordinate vectors we found in part d. $\left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$.

5. (10 points) Let $V$ and $W$ be two vector spaces and $T : V \to W$ a linear transformation from $V$ to $W$. Let $p$ be a positive integer and $\{f_1, \ldots, f_p\}$ a linearly dependent subset of $V$ consisting of $p$ elements. Show the the subset $\{T(f_1), \ldots, T(f_p)\}$ of $W$ is linearly dependent as well. Note: Provide an argument that works for general vector spaces, starting with the definition of linear dependence.

**Answer:** The set $\{f_1, \ldots, f_p\}$ is linearly dependent, if the equation $0 = c_1 f_1 + \cdots + c_p f_p$, with the scalar coefficients $c_i$ as unknowns, has a solution with at least one non-zero $c_i$. Choose such a solution and apply $T$ to both sides of the equation to get: 

$0 = T(0) = T(c_1 f_1 + \cdots + c_p f_p) = c_1 T(f_1) + \cdots + c_p T(f_p)$,

where in the first and last equalities we used the linearity properties of $T$. We conclude that the equation $0 = c_1 T(f_1) + \cdots + c_p T(f_p)$ has a solution with at least one non-zero $c_i$. Hence, the subset $\{T(f_1), \ldots, T(f_p)\}$ of $W$ is linearly dependent.
6. (20 points) Let \( C^\infty(\mathbb{R}) \) be the vector space of functions from \( \mathbb{R} \) to \( \mathbb{R} \), having derivatives of all orders. Denote by \( f \) all orders. Denote by \( T : V \to \mathbb{R}^3 \) be the transformation given by \( T(f) := \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} \).

(a) Show that the transformation \( T \) is linear. In other words, verify the following identities, for any two elements \( f, g \) of \( V \), and for every scalar \( k \).

i. \( T(f + g) = T(f) + T(g) \). \textbf{Answer:} \( T(f + g) = \begin{pmatrix} (f + g)(0) \\ (f + g)'(0) \\ (f + g)''(0) \end{pmatrix} = \begin{pmatrix} f(0) + g(0) \\ f'(0) + g'(0) \\ f''(0) + g''(0) \end{pmatrix} = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} + \begin{pmatrix} g(0) \\ g'(0) \\ g''(0) \end{pmatrix} = T(f) + T(g) \).

ii. \( T(kf) = kT(f) \). \textbf{Answer:} \( T(kf) = \begin{pmatrix} kf(0) \\ kf'(0) \\ kf''(0) \end{pmatrix} = k \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} = kT(f) \).

(b) Show that the subset \( \{T(f_1), T(f_2), T(f_3)\} \) of \( \mathbb{R}^3 \) is linearly independent. Hint: Recall that the chain rule yields \( (e^{2x})' = 2e^{2x}, (e^{2x})'' = 2^2e^{2x} \), and so \( f''(0) = 4 \).

\textbf{Answer:} \( T(f_1) = \begin{pmatrix} e^0 \\ e^0 \\ e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, T(f_2) = \begin{pmatrix} e^0 \\ 2e^0 \\ 4e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, T(f_3) = \begin{pmatrix} e^0 \\ 3e^0 \\ 9e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \).

Row reducing, we get:

\( (T(f_1)T(f_2)T(f_3)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \).

We get a pivot in every column, so the columns \( T(f_1), T(f_2), T(f_3) \) are linearly independent, and a pivot in every row, so \( T(f_1), T(f_2), T(f_3) \) span the whole of \( \mathbb{R}^3 \).

(c) Show that \( \text{im}(T) \) is the whole of \( \mathbb{R}^3 \).

\textbf{Answer:} \( \text{im}(T) = \text{span}\{T(f_1), T(f_2), T(f_3)\} \), and the latter was shown to be the whole of \( \mathbb{R}^3 \) in the previous part.

(d) Show the the subset \( \{e^x, e^{2x}, e^{3x}\} \) of \( V \) is linearly independent. Hint: Use part 6b and question 5.

\textbf{Answer:} We argue as in question 5. Suppose \( c_1e^x + c_2e^{2x} + c_3e^{3x} = 0 \). Applying \( T \) to both sides we get

\[ c_1T(e^x) + c_2T(e^{2x}) + c_3T(e^{3x}) = 0. \]

The vectors \( T(e^x), T(e^{2x}), T(e^{3x}) \) in \( \mathbb{R}^3 \) are linearly independent, by part 6b. Hence, \( c_1 = c_2 = c_3 = 0 \). Hence, the subset \( \{e^x, e^{2x}, e^{3x}\} \) of \( V \) is linearly independent.

(e) Show that \( T : V \to \mathbb{R}^3 \) is an isomorphism.

\textbf{Answer:} It suffices to show that \( \ker(T) = \{0\} \) and \( \text{im}(T) = \mathbb{R}^3 \). The equality \( \ker(T) = \mathbb{R}^3 \) was shown in part 6c. The Rank-Nullity-Theorem yields the equality \( \dim(\ker(T)) + \dim(\text{im}(T)) = \dim(V) \). The set \( \{e^x, e^{2x}, e^{3x}\} \) is linearly independent, by part 6d, and spans \( V \), by definition of \( V \), and is thus a basis for \( V \). The vector space \( V \) is three-dimensional, having a basis consisting of three elements. Hence, \( \dim(\ker(T)) = \dim(V) - \dim(\text{im}(T)) = 3 - 3 = 0 \). Thus, \( \ker(T) = \{0\} \).