Math 235 section 1 Solution for Midterm 1 Spring 2008

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system
$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 4$
$x_2 - x_3 + x_4 + x_5 = 4$
$2x_1 + 4x_3 + 3x_4 + 5x_5 = 2$
is $\begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$. Use at most five elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

Answer: Add $-2R_1$ to R_3, Add $2R_2$ to R_3, Add $-R_3$ to R_2, Add $-2R_3$ to R_1, Add $-R_2$ to R_1.

b) Find the general solution for the system.

Answer: The free variables are x_3 and x_5.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

2. (20 points) You are given that the row reduced echelon form of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 & 0 & 4 \\ 1 & 0 & 2 & -1 & 2 & 0 \end{pmatrix}$ is $B = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$. You do **not** need to verify this statement.

(a) Write the general solutions of the system $A\vec{x} = \vec{0}$ in parametric form $\vec{x} = (\text{first free variable})\vec{v}_1 + (\text{second free variable})\vec{v}_2 + \ldots$

Answer: The free variables are x_3 and x_6.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_3 + 2x_6 \\ x_3 \\ x_3 \\ -2x_6 \\ -2x_6 \\ x_6 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}.$$

(b) Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear transformations given by $T(\vec{x}) = A\vec{x}$. Find a basis for the kernel $\ker(T)$. In other words, find a linearly independent set of vectors in $\ker(T)$, which spans $\ker(T)$. Explain why the set you found is linearly independent, and why it spans $\ker(T)$.

Answer: $\ker(T)$ is the set of solutions of the equation $A\vec{x} = \vec{0}$. The vectors

$$v_1 := \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } v_2 := \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$ form a basis for $\ker(T)$. We have seen in part 2a that these two vectors span $\ker(T)$. These two vectors are clearly linearly independent, since none of them is a scalar multiple of the other. Hence, they are a basis.

(c) For each vector in the basis you found in part 2b, write down a corresponding linear relation among the columns of the original matrix A (use the notation a_i for the i-th column). Then use each of these relations to find a redundent vector among the columns of A (i.e., a column vector $\vec{a}_i$, which is a linear combination of the preceeding columns $\vec{a}_1, \ldots, \vec{a}_{i-1}$).

Answer: The equation $Av_1 = 0$ yields $-2\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = \vec{0}$, so $\vec{a}_3 = 2\vec{a}_2 - \vec{a}_2$, and $\vec{a}_3$ is redundent.

The equation $Av_2 = 0$ yields $2\vec{a}_1 - 2\vec{a}_4 - 2\vec{a}_5 + \vec{a}_6 = 0$, so $\vec{a}_6 = -2\vec{a}_1 + 2\vec{a}_4 + 2\vec{a}_5$, and $\vec{a}_6$ is redundent.

(d) Is the image of T equal to the whole of $\mathbb{R}^4$? Justify your answer.

Answer: Yes! The image of T is the set of values of T, i.e., the set of vectors $\vec{y}$ in $\mathbb{R}^4$ that can be written in the form $A\vec{x}$ for some vector $\vec{x}$ in $\mathbb{R}^6$. The row reduced echelon form of A has a pivot in every row, so the system $A\vec{x} = \vec{y}$ has a solution $\vec{x}$, for every vector $\vec{y}$.

3. (a) (7 points) Let A, B, C be $n \times n$ matrices, with A invertible, which satisfy the equation $A(C + I_n)A^{-1} = B$, where I_n is the $n \times n$ identity matrix. Express C in terms of A and B. Show all your work.

Answer: Multiply both sides by A^{-1} of the left and by A on the right to get $C + I_n = A^{-1}BA$. Thus, $C = A^{-1}BA - I_n$.

(b) (8 points) Let A be an $n \times n$ matrix satisfying $A^3 + 5A^2 + 2A - I_n = 0$, where 0 is the $n \times n$ matrix all of which entries are zero. Show that A is invertible and express A^{-1} in terms of A.

Hint: Rewrite the equation as $A^3 + 5A^2 + 2AI_n - I_n = 0$.

Answer: $A^3 + 5A^2 + 2AI_n = I_n$ and we can factor the left hand side to get $A(A^2 + 5A + 2I_n) = I_n$. Hence, $A^{-1} = A^2 + 5A + 2I_n$.

4. (15 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation with standard matrix A. Assume that there exists a unique vector $\vec{x}$ in $\mathbb{R}^3$, such that $T(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Carfully **justify** your answers to the following questions.

(a) The rank of A is: $\underline{\ 3\ }$.

Reason: A is a 4×3 matrix. The fact that the solution to the above equation is unique means that there aren't any free variables, so A has a pivot position in every column.

(b) Describe geometrically the kernel of T.

Answer: $\ker(T)$ is the set of solutions of the system $A\vec{x} = \vec{0}$. The zero vector is the unique solution, since A has a pivot position in every column. So $\ker(T)$ is a single point (the origin) in $\mathbb{R}^3$.

(c) Is it true that the equation $A\vec{x} = \vec{y}$ has a unique solution $\vec{x}$, for every vector $\vec{y}$ in $\mathbb{R}^4$? Justify!

Answer: This is false. For some choices of vectors $\vec{y}$ in $\mathbb{R}^4$ the equation $A\vec{x} = \vec{y}$ will be inconsistent, since A does not have a pivot in every row (there are 4 rows and only three pivots).

5. (a) (8 points) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$.

Answer: $A^{-1} = \begin{pmatrix} 4 & -2 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

(b) (7 points) Find the set of all matrices B, satisfying the matrix equation $BA = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

Answer: $B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 5 & -2 & -4 \\ 3 & -1 & -2 \end{pmatrix}$. We have found the unique such matrix.

6. (20 points + 9 **bonus** points, you got 3 extra points for a correct answer to each of parts 6a, 6b, 6c below) Let L_θ be the line in $\mathbb{R}^2$ through the origin and the unit vector $\vec{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Recal that the reflection $Ref_{L_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane about the line L_θ is given by the formula $Ref_{L_\theta}(\vec{x}) = 2\,(\vec{u} \cdot \vec{x})\,\vec{u} - \vec{x}$.

(a) Use the algebraic properties of the dot product to show that Ref_{L_θ} is a linear transformation. In other words, verify the following identities, for any two vectors $\vec{v}, \vec{w}$ and for every scalar k.

i. $Ref_{L_\theta}(\vec{v} + \vec{w}) = Ref_{L_\theta}(\vec{v}) + Ref_{L_\theta}(\vec{w})$.
Answer: $Ref_{L_\theta}(\vec{v} + \vec{w}) = 2\,(\vec{u} \cdot [\vec{v} + \vec{w}])\,\vec{u} - [\vec{v} + \vec{w}] =$
$2\,(\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w})\,\vec{u} - [\vec{v} + \vec{w}] = [2\,(\vec{u} \cdot \vec{v}) - \vec{v}] + [2\,(\vec{u} \cdot \vec{w}) - \vec{w}] =$
$Ref_{L_\theta}(\vec{v}) + Ref_{L_\theta}(\vec{w})$.

ii. $Ref_{L_\theta}(k\vec{v}) = kRef_{L_\theta}(\vec{v})$.
Answer: $Ref_{L_\theta}(k\vec{v}) = 2\,(\vec{u} \cdot (k\vec{v}))\,\vec{u} - k\vec{v} = k\,[2\,(\vec{u} \cdot \vec{v})\,\vec{u} - \vec{v}] = kRef_{L_\theta}(\vec{v})$.

(b) Use the above formula for $Ref_{L_\theta}(\vec{x})$ to show that the standard matrix A of Ref_{L_θ} is $A = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$. Hint: Recall the identities:
$\cos^2(\theta) + \sin^2(\theta) = 1$, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$.

Answer: Let $\vec{a}_1$ and $\vec{a}_2$ be the two columns of A. Then
$\vec{a}_1 = Ref_{L_\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$

$$\begin{pmatrix} 2\cos^2(\theta)-1 \\ 2\cos(\theta)\sin(\theta) \end{pmatrix} = \begin{pmatrix} 2\cos^2(\theta)-\cos^2(\theta)-\sin^2(\theta) \\ 2\cos(\theta)\sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}.$$

$$\vec{a}_2 = Ref_{L_\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\left(\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 2\sin(\theta)\cos(\theta) \\ 2\sin^2(\theta)-1 \end{pmatrix} = \begin{pmatrix} \sin(2\theta) \\ \sin^2(\theta)-\cos^2(\theta) \end{pmatrix} = \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}.$$

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition $T(\vec{x}) = Ref_{L_\phi}(Ref_{L_\theta}(\vec{x}))$, where L_ϕ is the line through the origin and $\vec{w} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix}$. Express the standard matrix C of T in terms of the standard matrices A of Ref_{L_θ} and B of Ref_{L_ϕ}: $C = \underline{\quad BA \quad}$.

Use this expression to show the equality $C = \begin{pmatrix} \cos(2\phi-2\theta) & -\sin(2\phi-2\theta) \\ \sin(2\phi-2\theta) & \cos(2\phi-2\theta) \end{pmatrix}$.

Hint: Recall the identities $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ and $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$.

Answer: $C = BA = \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} =$

$$\begin{pmatrix} [\cos(2\phi)\cos(2\theta)+\sin(2\phi)\sin(2\theta)] & [\cos(2\phi)\sin(2\theta)-\sin(2\phi)\cos(2\theta)] \\ [\sin(2\phi)\cos(2\theta)-\cos(2\phi)\sin(2\theta)] & [\sin(2\phi)\sin(2\theta)+\cos(2\phi)\cos(2\theta)] \end{pmatrix} =$$

$$\begin{pmatrix} \cos(2\phi-2\theta) & -\sin(2\phi-2\theta) \\ \sin(2\phi-2\theta) & \cos(2\phi-2\theta) \end{pmatrix}.$$

(d) The linear transformation T in part 6c is described more directly as the rotation of the plane about the origin an angle $\underline{\quad 2\phi-2\theta \quad}$ counterclockwise. **Justify** your answer.

Reason: We have seen in class, that the matrix of a rotation of the plane by angle α counterclockwise is given by the matrix $\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$. Part 6c shows that $\alpha = 2\phi-2\theta$.