Math 235 Section 1 Solution of Final Exam Spring 2008

Justify all your answers. Show all your work!!!

1. (10 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for $\ker(A)$.

b) Find a basis for image(A).

The solution is similar to question 1 in midterm 2.

2. (16 points) Consider the matrix $A = \begin{pmatrix} -1 & -2 & -4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}$.

(a) Show that the characteristic polynomial of A is $-(\lambda-1)(\lambda+1)(\lambda-2)$.

Answer: $\det(A-\lambda I) = \det \begin{pmatrix} -1-\lambda & -2 & -4 \\ 0 & -\lambda & -1 \\ 0 & 2 & 3-\lambda \end{pmatrix}$. Now use the co-factor (Laplace) expansion along the first column to get $\det(A-\lambda I) = -(\lambda+1)[\lambda^2 - 3\lambda + 2] = -(\lambda-1)(\lambda+1)(\lambda-2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Answer: The eigenvalues are the roots of the characteristoc polynomial $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

<u>The 1-eigenspace:</u> $\ker(A - 1I) = \ker(A-I) = \ker \begin{pmatrix} -2 & -2 & -4 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}$. Row reducing, we get that the row reduced echelon form of $A-I$ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

We see that x_3 is a free variable, $x_1 = -x_3$, $x_2 = -x_3$, and so the general vector in $\ker(A-I)$ has the from $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. The 1-eigenspace is thus spanned by $v_1 := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

<u>The −1-eigenspace:</u> $\ker(A-(-1)I) = \ker(A+I)$ is shown similarly to be spanned by $v_2 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The 2-eigenspace: $\ker(A-2I)$ is shown similarly to be spanned by $v_3 := \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

(c) Find an invertible matrix S and a diagonal matrix D such that the matrix A above satisfies $S^{-1}AS = D$

Answer: A theorem we prove in class tells us that if we take the union of the bases for all the eigenspaces, we get a linearly independent set. The set $\{v_1, v_2, v_3\}$ is thus linearly independent, and since it consists of three vectors, it is a basis of $\mathbb{R}^3$.

The Diagonalization Theorem states that the matrix A is diagonalizable, if and only if there exists a basis of R^3 consisting of eigenvectors of A. Furthermore, if $\{v_1, v_2, v_3\}$ is such a basis, and we let $S = (v_1 v_2 v_2)$ be the matrix, whose j-th column is v_j, then $S^{-1}AS$ is a diagonal matrix, whose (j,j) diagonal entry is the eigen-value λ_j of v_j.

We can thus take $S = \begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

3. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

(a) The eigenvalue of v_1 is $\underline{\ 1\ }$, since $Av_1 = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$.

The eigenvalue of v_2 is $\underline{\ 0.4\ }$, since $Av_2 = 0.4v_2$.

(b) Set $w := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the coordinate vector $[w]_\beta$ of w in the basis $\beta := \{v_1, v_2\}$.

Answer: Row reduce $(v_1 v_2 | w) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \end{pmatrix}$.
We see that $w = (3/2)v_1 - (1/2)v_2$.

(c) Compute $A^{100}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Answer: $A^{100}w = A^{100}[(3/2)v_1 - (1/2)v_2] = 1^{100}(3/2)v_1 - (0.4)^{100}(1/2)v_2 = \begin{pmatrix} (3/2) - (1/2)(.4)^{100} \\ (3/2) + (1/2)(.4)^{100} \end{pmatrix}$.

(d) As n gets larger, the vector $A^n\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches $\underline{\ (3/2)v_1\ }$. Justify your answer.

Answer: Using the fact that $\lim_{n\to\infty}(.4)^n = 0$, we get
$A^n w = A^n[(3/2)v_1 - (1/2)v_2] = (3/2)v_1 - (0.4)^n(1/2)v_2 \overset{n\to\infty}{\longrightarrow} (3/2)v_1 = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$.

4. (16 points) Let V be the plane in $\mathbb{R}^3$ spanned by $v_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $v_2 := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) Find the orthogonal projection $\mathrm{proj}_V(w)$ of $w = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ into V.

Answer: We check first that $\beta := \{v_1, v_2\}$ is an orthogonal basis for V. The set β spans V, by definition of V, and it consists of non-zero vectors, so it suffices to check that the set is orthogonal. Indeed, $v_1 \cdot v_2 = 0$. We can now apply the formula for the projection

$$\mathrm{proj}_V(w) = \frac{(w \cdot v_1)}{(v_1 \cdot v_1)} v_1 + \frac{(w \cdot v_2)}{(v_2 \cdot v_2)} v_2 = 3v_1 + 2v_2 = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

(b) Write w as a sum of a vector in V and a vector orthogonal to V.

Answer: By definition of the projection to V, $\mathrm{proj}_V(w)$ is the vector in V, such that $w - \mathrm{proj}_V(w)$ is orthogonal to V, and in particular to $\mathrm{proj}_V(w)$. Thus, $w = \mathrm{proj}_V(w) + [w - \mathrm{proj}_V(w)] = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ is such a sum.

(c) Find the distance from w to V (i.e., to the vector in V closest to w).

Answer: The point in V closest to w is $\mathrm{proj}_V(w)$. Thus the distence from w to V is $\|w - \mathrm{proj}_V(w)\| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{8}$.

5. (16 points)

(a) Let A and S be two $n \times n$ matrices with real coefficients with S invertible. Then the columns $v_1, \ldots, v_n$ of S form a basis of $\mathbb{R}^n$. Complete the following sentence: The matrix $S^{-1}AS$ is diagonal with d_i as its (i,i)-entry, if and only if for all $1 \leq i \leq n$, the vector v_i is

<u>*an eigen-vector of A with eigen-value d_i.*</u>.

(b) For what values of θ is the matrix $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ diagonalizable? I.e., for what values of θ does there exist some invertible 2×2 matrix S with real coefficients, such that $S^{-1}AS$ is diagonal? Justify your answer!

Answer: <u>Method A:</u> The characteristic polynomial of A is $\det \begin{pmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{pmatrix} = \lambda^2 - 2\cos(\theta)\lambda + 1$. Its discriminat $4\cos^2(\theta) - 4$ is negative, unless $\cos(\theta) = \pm 1$, i.e., $\theta = n\pi$, for some integer n. If $\theta = n\pi$, then $A = \pm I$ is diagonal. Otherwise, the characteristic polynomial does not have any real root, so A does not have any real eigenvalues, and is thus not similar to a diagonal matrix with real entries (not diagonalizable).

<u>Method B:</u> We can also argue geometrically and arrive at the same conclusion. I.e., A is diagonalizable, if and only if it is rotation by angle 0 (the identity) or by andle π (so $A = -I$), since otherwise for every non-zero vector v in $\mathbb{R}^2$, v and Av do not lie on a line through the origin, so Av is not a scalar multiple of v (so A does not have any eigen-vectors).

(c) For what values of k is the matrix $\begin{pmatrix} 2 & 0 \\ k & 2 \end{pmatrix}$ diagonalizable? Justify your answer!

Answer: The matrix is lower triangular, so its eigen-values are its diagonal entries. In our case, the only eigen-value is 2. A is diagonalizable,
if and only if the 2-eigen-space is 2-dimensional,
if and only if $\ker(A - 2I)$ is 2-dimensional,
if and only if $\ker \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$ is 2-dimensional,
if and only if $\operatorname{rank} \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} = 0$,
if and only if $k = 0$.

6. (10 points)

Let V be the subspace of $\mathbb{R}^4$ spanned by $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \\ -1 \end{pmatrix}$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for V.
Answer: Take $u_1 := \frac{1}{\|v_1\|} v_1$, $\tilde{u}_2 := v_2 - \operatorname{proj}_{v_1}(v_2)$, and $u_2 := \frac{1}{\|\tilde{u}_2\|} \tilde{u}_2$. Then $\{u_1, u_2\}$ is an orthonormal basis for V. Calculating, we get:

$$u_1 = \tfrac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \ \operatorname{proj}_{v_1}(v_2) = \tfrac{(v_1 \cdot v_2)}{(v_1 \cdot v_1)} v_1 = 2v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ -2 \end{pmatrix},$$

$$\tilde{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \ u_2 = \tfrac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Find a basis for the orthogonal complement $V^\perp$ of V in R^4.

Answer: The orthogonal complement $V^\perp$ of V is the kernel of $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 3 & 1 & 3 & -1 \end{pmatrix}$.

Row reducing, we find that $w_1 := \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $w_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $V^\perp$.

7. (16 points) Let P_3 be the vector space of polynomials of degree ≤ 3 with real coefficients. Let $T : P_3 \to \mathbb{R}^4$ be the linear transformation given by

$T(f) = \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{pmatrix}$. Consider the following four polynomials in P_3:

$f_1(x) = \frac{-1}{6}(x-2)(x-3)(x-4)$, $\quad f_2(x) = \frac{1}{2}(x-1)(x-3)(x-4)$,
$f_3(x) = \frac{-1}{2}(x-1)(x-2)(x-4)$, $\quad f_4(x) = \frac{1}{6}(x-1)(x-2)(x-3)$.

Let $U : \mathbb{R}^4 \to P_3$ be the linear transformation given by

$$U\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x).$$

(a) Show that the composition $TU : \mathbb{R}^4 \to \mathbb{R}^4$ is the identity linear transformation. In other words, show that $T(U(\vec{x})) = \vec{x}$, for all $\vec{x}$ in $\mathbb{R}^4$.

Answer: A straightforward calculation shows that $T(f_i) = e_i$, where e_i is the i-th column of the 4×4 identity matrix. For example $T(f_1) = \begin{pmatrix} f_1(1) \\ f_1(2) \\ f_1(3) \\ f_1(4) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Thus, $T\left(U\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_3 \end{pmatrix}\right) =$

$$T(c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4) = c_1 T(f_1) + c_2 T(f_2) + c_3 T(f_3) + c_4 T(f_4) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

(b) Show that T is an isomorphism. Hint: Show first that image$(T) = \mathbb{R}^4$.

Answer: Given a vector $\vec{x}$ in $\mathbb{R}^4$, we have $T(U(\vec{x})) = \vec{x}$, by the previous part, and so $\vec{x}$ is a value of T, and $\text{im}(T) = \mathbb{R}^4$.

T is an isomorphism, if and only if $\ker(T) = 0$ and $\text{im}(T) = \mathbb{R}^4$ (which we have already established). The rank-nullity theorem states that $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(P_3)$. In our case, the equation becomes $\dim(\ker(T)) + 4 = 4$, so $\ker(T) = 0$.

(c) Show that the set $\{f_1, f_2, f_3, f_4\}$ is a basis of P_3. Use the previous parts to minimize your calculations.

Answer: We have seen that T is an isomorphism, and U is its inverse. Thus, U is an isomorphism as well. An isomorphism maps a basis to a basis. U takes the standard basis to $\{f_1, f_2, f_3, f_4\}$. Hence, the latter set is a basis.

Note: Once we know that $\beta := \{f_1, f_2, f_3, f_4\}$ is a basis of P_3, then part 7a means that T is nothing but the linear transformation sending a polynomial f in P_3 to its coordinate vector $[f]_\beta$.

(d) Find a polynomial $g(x)$ of degree ≤ 3 satisfying $g(1) = 2$, $g(2) = 3$, $g(3) = 5$, $g(4) = 7$. Hint: Express g as a linear combination of the f_i's. You need not simplify your answer.

Answer: We are looking for a polynomial $g(x)$ satisfying $T(g) = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$. Take

$g(x) = 2f_1(x) + 3f_2(x) + 5f_3(x) + 7f_4(x)$.