TEST 2:Answers

1. (*2 points*) What was Gauss's first name?
Carl

2. (*15 points*) Which of the following sets are subspaces of vector space $\mathcal{P}_5$?

Set	Subspace	Not a subspace
$\{p \in \mathcal{P}_5 \mid p'(0) > 0\}$		$\surd$
$\{p \in \mathcal{P}_5 \mid p'(0) = 0\}$	$\surd$	
$\{p \in \mathcal{P}_5 \mid p'(4) = 0\}$	$\surd$	
$\{p \in \mathcal{P}_5 \mid p'(0) = 4\}$		$\surd$
My other dog Pepsi		$\surd$

3. (*18 points*) Suppose that

- O and N are an orthogonal matrices.
- S and T are symmetric matrices.
- E and W are skew-symmetric matrices.

Write *True* for the statements that MUST be true, and write *False* otherwise.

a. (*2 pts*) <u>*True*</u> W is a square matrix.

b. (*2 pts*) <u>*False*</u> E is an invertible matrix.

c. (*2 pts*) <u>*True*</u> S is a square matrix.

d. (*2 pts*) <u>*False*</u> T is an invertible matrix.

e. (*2 pts*) <u>*True*</u> O is a square matrix.

f. (*2 pts*) <u>*True*</u> N is an invertible matrix

g. (*2 pts*) <u>*False*</u> SON is an orthogonal matrix.

h. (*2 pts*) <u>*True*</u> $ST - TS$ is a skew-symmetric matrix.

i. (*2 pts*) <u>*True*</u> EE^t is a symmetric matrix.

4. (*14 points*) Let V be the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 7 \\ -8 \end{pmatrix}$.

a. (*7 pts*) Find an orthonormal basis of V.

Use Gram-Schmidt:

$$\vec{u}_1 = \vec{v}_1/\|\vec{v}_1\| = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}.$$

$$\begin{aligned} \vec{u}_2 &= (\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1)/\|\vec{v}_2^{\perp}\| \\ &= \frac{1}{\|\vec{v}_2^{\perp}\|}\left(\begin{pmatrix} 2 \\ 7 \\ -8 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right) \quad \text{since } \vec{v}_2 \cdot \vec{u}_1 = 9. \\ &= \frac{1}{\|\vec{v}_2^{\perp}\|}\begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2/3 \\ 2/3 \\ -1/3 \end{pmatrix} \quad \text{since } \|\vec{v}_2^{\perp}\| = 6. \end{aligned}$$

b. (*7 pts*) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto V.

Recall that

$$\text{proj}_V(\vec{v}) = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2.$$

It follows that the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto V is

$$\frac{2}{3}\vec{u}_1 - \frac{2}{3}\vec{u}_2 = \begin{pmatrix} 4/9 \\ 2/9 \\ -4/9 \end{pmatrix} + \begin{pmatrix} 4/9 \\ -4/9 \\ 2/9 \end{pmatrix} = \begin{pmatrix} 8/9 \\ -2/9 \\ -2/9 \end{pmatrix}.$$

5. (*15 points*) Let $\mathbb{C}$ denote the *complex numbers* with the basis $\mathcal{B} = (1, i)$. Let T be the linear transformations $T(z) = (3+4i)z$ from $\mathbb{C}$ to $\mathbb{C}$.

a. (*7 pts*) Find the matrix of T in the basis $\mathcal{B}$. i.e. Compute $A = \text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T)$.

$$A = \text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T) = \big[[T(1)]_{\mathcal{B}} \quad [T(i)]_{\mathcal{B}}\big] = \big[[3+4i]_{\mathcal{B}} \quad [3i-4]_{\mathcal{B}}\big] = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

b. (*8 pts*) Compute the QR factorization of A.

Note that the columns of A are perpendicular to each other and length 5. It follows that the QR-factorization of A has

$$Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

6. (*20 points*) Let $\mathbb{R}^{2\times 2}$ denote the 2×2 matrices. There is a map called *trace*, $\text{Tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$, that is defined by

$$\text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

a. (*3 pts*) What is the dimension of $\mathbb{R}^{2\times 2}$?
Let E_{ij} be the 2×2 matrix with a 1 in the (i, j) slot and 0 everywhere else. The dimension of $\mathbb{R}^{2\times 2}$ is seen to be 4, since $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ form a basis.

b. (*6 pts*) Verify that Tr is a linear transformation.
We need to check that Tr satisfies three conditions.

1. $\text{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 + 0 = 0.$
2. $\text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix}\right) = \text{Tr}\left(\begin{bmatrix} a+\hat{a} & b+\hat{b} \\ c+\hat{c} & d+\hat{d} \end{bmatrix}\right) = a+\hat{a}+d+\hat{d} = \text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \text{Tr}\left(\begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix}\right).$
3. $\text{Tr}\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \text{Tr}\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka + kd = k(a + d) = k\,\text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$

c. (*6 pts*) A matrix A is said to be *traceless* if $\text{Tr}(A) = 0$. Let S denote the subspace of traceless 2×2 matrices. What is the dimension of S? Find a basis for S.
Notice that $S = \ker(\text{Tr})$. Since $\text{im}(\text{Tr})$ is 1 dimensional and $\mathbb{R}^{2\times 2}$ is 4 dimensional, the Rank-Nullity Theorem tells us that the dimension of S is 3. A basis of S is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

d. (*5 pts*) Consider the linear transformation $Q : S \to \mathbb{R}^2$ defined by

$$Q(M) = M\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the kernel of Q.
The kernel of Q is the set of traceless matrices A such that $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} a \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, $\ker(Q) = \left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \;\middle|\; b \in \mathbb{R}\right\}$. Equivalently, $\ker(Q) = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$.

7. (*16 points*) Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

a. (*12 pts*) Find the least squares solutions of $A\vec{x} = \vec{b}$.
Recall that the least squares solutions to $A\vec{x} = \vec{b}$ are the solutions to the normal equation $A^tA\vec{x} = A^t\vec{b}$. Thus we want to solve

$$A^tA\vec{x} = A^t\vec{b}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We use Gaussian elimination on $\begin{bmatrix} 2 & 1 & 2 & : & 1 \\ 1 & 2 & 1 & : & 0 \\ 2 & 1 & 2 & : & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 1 & : & 2/3 \\ 0 & 1 & 0 & : & -1/3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$. It follows that the least squares solutions are

$$\begin{pmatrix} 2/3 - t \\ -1/3 \\ t \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

Equivalently, the least squares solutions are the line

$$\begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

b. (*4 pts*) Find a point in the image of A that is as close to $\vec{b}$ as possible.
Recall that the least squares solution is the solution to $A\vec{x} = \text{proj}_{\text{im}(A)}(\vec{b})$. In other words, the least squares solutions represent the solutions to

$$A\vec{x} = \text{"as close to } \vec{b} \text{ as we can get and still have a solution."}\ .$$

It follows that $A \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ -1/3 \end{pmatrix}$ is as close to $\vec{b}$ as we can get.

BONUS Survey: 1 bonus point total for completion

1. $327 \times 151 = 49,377$

2. Out of 100 points, what do you think is your score on this test? 95

3. Out of 100 points, what do you think is the class average on this test? 75