Practice Problems

1. (*8 points*) Which of the following subsets $S \subseteq V$ are subspaces of V? Write *YES* if S is a subspace and *NO* if S is not a subspace.

a. (*2 pts*) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \leq y \leq z \right\}$

b. (*2 pts*) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$

c. (*2 pts*) S is the set of vectors of the form $\begin{pmatrix} a + 2b + 3c \\ c \\ 0 \end{pmatrix}$.

d. (*2 pts*) S is the set polynomials in $\mathcal{P}_3$ such that $p'(2) = 0$.

2. (*10 points*) Solve the following system of linear equations.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

3. (*10 points*) Solve the following system of linear equations.

$$\begin{aligned} x - z &= 1 \\ x + 2y + 3z &= 11. \end{aligned}$$

4. (*7 points*) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation counterclockwise about the origin in $\mathbb{R}^2$ by $\frac{\pi}{4}$ radians or $45°$.

a. (*3 pts*) Compute the matrix that represents T.
b. (*2 pts*) Is T an isomorphism?
c. (*2 pts*) Is T diagonalizable?

5. (*6 points*) Let A be a $n \times n$ orthogonal matrix.

a. (*2 pts*) What is the rank of A?
b. (*2 pts*) What are the possible values for $\det(A)$?
c. (*2 pts*) If λ is an eigenvalue for A, what are the possible values for λ?

6. (*13 points*) Consider the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $(T(f))(x) = f(2x - 1)$. Let $\mathcal{B}$ be the ordered basis $\mathcal{B} = (1, x, x^2)$.

a. (*3 pts*) Compute $\text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T)$.
b. (*3 pts*) Compute the eigenvalues of T.
c. (*3 pts*) Is T diagonalizable?
d. (*4 pts*) Compute the eigenspaces of T. Make sure your answers are expressed as subspaces of $\mathcal{P}_2$.

7. (*12 points*) Two interacting populations of foxes and hares can be modeled by the equations

$$h(t+1) = 4h(t) - 2f(t)$$
$$f(t+1) = h(t) + f(t).$$

a. (*4 pts*) Find a matrix A such that

$$\begin{pmatrix} h(t+1) \\ f(t+1) \end{pmatrix} = A \begin{pmatrix} h(t) \\ f(t) \end{pmatrix}.$$

b. (*8 pts*) Find a formula for $h(t)$ and $f(t)$.

8. (*10 points*) Let A be a 3×3 matrix such that

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ as solutions. Find another solution. Explain.

9. (*12 points*) Let $T : \mathbb{R}^{9\times 10} \to \mathbb{R}^9$ be the map defined by

$$T(A) = A\vec{e}_1.$$

a. (*4 pts*) Show that T is a linear transformation.
b. (*4 pts*) What is the rank of T?
c. (*4 pts*) State the Rank-Nullity Theorem and use it to compute the nullity of T.

10. (*12 points*)
a. (*4 pts*) Give the definition of the phrase V *is a subspace of* $\mathbb{R}^n$.
b. (*8 pts*) Let V be a subspace of $\mathbb{R}^n$. Prove that $V^\perp = \{\vec{u} \in \mathbb{R}^n \mid \vec{u}\cdot\vec{v} = 0 \quad \text{for every } \vec{v} in V\}$ is a subspace of $\mathbb{R}^n$.