Practice TEST 2: hints

1. (*20 points*) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and let $\vec{v}_2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Let V be the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$.

a. (*5 pts*) Prove that $\vec{v}_1$ is not perpendicular to $\vec{v}_2$.
Compute the dot product $\vec{v}_1 \cdot \vec{v}_2$.

b. (*8 pts*) Find an orthonormal basis for V.
Use Gram-Schmidt on $\{\vec{v}_1, \vec{v}_2\}$ to find orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$.

c. (*7 pts*) What is the matrix for orthogonal projection onto V?
Compute QQ^t, where $Q = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}$.

2. (*17 points*) Find the quadratic polynomial $p(t) = a + bt + ct^2$ that best (in the least squares sense) fits the following data.

t	-1	0	1	2
y	1	1.5	2	3

If we were able to find a quadratic polynomial that went through all four points, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ would be a solution to $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 1.5 \\ 2 \\ 3 \end{pmatrix}.$$

(Why?) This has no solution, but we can find the least squares solution by solving the normal equation for this linear system,

$$A^t A\vec{x} = A^t\vec{b}.$$

The last step is to translate the solution back to a polynomial.

3. (*28 points*) Let $V \subseteq C^\infty$ be subspace spanned by $\{e^x, xe^x, x^2e^x\}$. Let $\mathcal{B}$ be the ordered basis

$$\mathcal{B} = (e^x, xe^x, x^2e^x).$$

a. (*4 pts*) What is the dimension of V?
The dimension is the number of elements in a basis.

b. (*8 pts*) Let $D : V \to V$ be the linear transformation given by $D(f) = f'$. Express D as a matrix with respect to the basis $\mathcal{B}$. i.e. Compute $\mathrm{Mat}_{\mathcal{B}}^{\mathcal{B}}(D)$.

$\mathrm{Mat}_{\mathcal{B}}^{\mathcal{B}}(D)$ is given by $\begin{bmatrix} | & | & | \\ [D(e^x)]_{\mathcal{B}} & [D(xe^x)]_{\mathcal{B}} & [D(x^2e^x)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$. Compute this 3×3 matrix.

c. (*8 pts*) Let $A = \mathrm{Mat}_{\mathcal{B}}^{\mathcal{B}}(D)$. You can check that

$$A^3 - 3A^2 + 3A - 1 = 0.$$

Consider the function $f(x) = 2e^x - 13xe^x + \sqrt{2}x^2e^x$. What does the above tell you about

$$f''' - 3f'' + 3f' - f?$$

The above tells us that the linear transformation $T = D^3 - 3D^2 + 3D - 1$ is the zero transformation on V. Hence $T(g) = 0$ for every $g \in V$. In particular, since $f \in V$,

$$T(f) = f''' - 3f'' + 3f' - f = 0.$$

d. (*8 pts*) Suppose you want to find functions u such that

$$u'''(x) - 3u''(x) + 3u'(x) - u(x) = x.$$

Verify that $u_0(x) = -x - 3$ is a solution. Find another one.
To verify u_0 is a solution, plug in and check.
Let h be the function $h(x) = x$. View T as a linear transformation $T : C^\infty \to C^\infty$. You are trying to find solutions to

$$T(u) = h.$$

You are given that $T(u_0) = h$. Notice that every other solution has the form $u_0 + \phi$, where $\phi \in \ker(T)$. (Why?) Do you know any non-zero things in $\ker(T)$?

4. (*15 points*) Find a basis for the space perpendicular to the solutions of

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 &= 0 \\ -2x_1 + 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

The solutions to those equations are exactly the kernel of $A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}$. Do you know another way to characterize $(\ker(A))^\perp$?

5. (*20 points*) Let P_5 denote the vector space of polynomials of degree at most 5. Let $S \subseteq P_5$ denote the subset of polynomials p such that

$$p''(2) = p(4).$$

Show that S is a subspace of P_5 and compute a basis of S.
To show that S is a subspace, verify the 3 conditions required to be a subspace.
A general element of P_5 can be written as $a + bx + cx^2 + dx^3 + ex^4 + fx^5$. What conditions are there on (a, b, c, d, e, f) so that the associated polynomial is in S? Set it up as a linear system and solve. Translate back to polynomials to get a basis for S.