Practice Problems Math 235 Spring 2007: Solutions

1. Write the system of equations as a matrix equation and find all solutions using Gauss elimination:
$$x + 2y + 4z = 0, -x + 3y + z = -5, 2x + y + 5z = 3.$$

   We see that this is a linear system with 3 equations in 3 unknowns. The matrix equation is $A\vec{x} = \vec{b}$, where
$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix}.$$

   To solve this system, we form the augmented matrix $\begin{bmatrix} 1 & 2 & 4 & : & 0 \\ -1 & 3 & 1 & : & -5 \\ 2 & 1 & 5 & : & 3 \end{bmatrix}$ and perform Gaussian elimination to get the coefficient matrix in reduced echelon form.
$$\begin{bmatrix} 1 & 2 & 4 & \vdots & 0 \\ -1 & 3 & 1 & \vdots & -5 \\ 2 & 1 & 5 & \vdots & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & \vdots & 0 \\ 0 & 5 & 5 & \vdots & -5 \\ 0 & -3 & -3 & \vdots & 3 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & -3 & -3 & \vdots & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

   Since there is no pivot in the third column, the third unknown is free and the solutions are $x = 2 - 2t, y = -1 - t, z = t$ or equivalently $\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, where t is free.

2. What does it mean for a vector to be in the kernel of a matrix A. Let A be the matrix $\begin{bmatrix} 1 & 2 & 5 \\ -2 & 0 & -2 \\ 3 & -1 & 1 \end{bmatrix}$. Is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ an element of the kernel of A? Why?

   A vector $\vec{v}$ is in the kernel of A if $A\vec{v} = \vec{0}$. To see that $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \in \ker(A)$, we compute
$$\begin{bmatrix} 1 & 2 & 5 \\ -2 & 0 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 + 4 - 5 \\ -2 + 0 + 2 \\ 3 - 2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Define what it means for a set s to be a basis of a subspace $V \subset \mathbb{R}^n$. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 4 & 3 & -5 \end{bmatrix}.$$

Give a set of vectors that span $\ker(A)$ and that are independent.

A set of vectors s is a basis of V is $V = \text{span}(s)$ and s is linearly independent. To find a basis for $\ker(A)$, we use Gaussian elimination to compute the reduced echelon form of A.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 4 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & 4 & -2 \\ 0 & 6 & 6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 6 & -6 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We see that the 4^{th} column has no pivot, and so x_4 is free. Then $\ker(A)$ is $x_1 = -t, x_2 = t, x_3 = 0, x_4 = t$ or equivalently $\vec{x} = t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Therefore a basis for $\ker(A)$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

4. Let A be a n by m matrix, so A gives a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^m$. Assume that $A(\vec{x}_1) = A(\vec{x}_2)$. Show that $\vec{x}_1 - \vec{x}_2$ is in the kernel of A.

We compute
$$A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{0}.$$

5. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a vector of length 1. Let A be a matrix whose effect on the plane is to reflect about the line through the origin and $\vec{u}$. Let $\vec{v} = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$. In terms of $\vec{u}$ and $\vec{v}$ what is $A\vec{u}$? what is $A\vec{v}$? Write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of $\vec{u}$ and $\vec{v}$. Use the answer to the previous question to compute $A\vec{e}_1$.

Notice that $\vec{v}$ is perpendicular to $\vec{u}$ (Check by computing dot product), and hence perpendicular to the line through the origin and $\vec{u}$. Since A is reflection about this line, it follows that $A\vec{v} = -v$. Since $\vec{u}$ is on the line, $A\vec{u} = \vec{u}$. We can use Gaussian elimination on $\begin{bmatrix} u_1 & -u_1 & \vdots & 1 \\ u_2 & u_1 & \vdots & 0 \end{bmatrix}$ to express $\vec{e}_1$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Alternatively, notice that $u_1\vec{u} - u_2\vec{v} = \begin{pmatrix} u_1^2 + u_2^2 \\ u_1u_2 - u_1u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, since $\vec{u}$ has length 1.

Now compute

$$\begin{aligned} A\vec{e}_1 &= A(u_1\vec{u} - u_2\vec{v}) \\ &= u_1\vec{u} + u_2\vec{v} \\ &= \begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{pmatrix}. \end{aligned}$$

6. Solve the equation

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by find the inverse of the given matrix.

7. Compute the product AB of the two matrices A, B given below, if possible. If it is not possible say why it is not possible.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 4 & 8 \end{bmatrix}$$

The product matrix AB gives a function. What is the domain and what is the range of that function?

Since B gives a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ and A gives a transformation $\mathbb{R}^2 \to \mathbb{R}^3$, only the composition AB makes sense and give a map from $\mathbb{R}^2 \to \mathbb{R}^3$. We compute the product

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 1 & 0 \\ -11 & -16 \end{bmatrix}$$

8. Find a basis of the subspace of $\mathbb{R}^3$ defined by $3x - y + z = 0$. What is the dimension of this subspace?

This subspace is the kernel of $A = \begin{bmatrix} 3 & -1 & 1 \end{bmatrix}$. Since $\text{rank}(A) = 1$, the Rank-Nullity Theorem implies that the dimension of $\ker(A) = 3 - 1 = 2$. By inspection, $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are in $\ker(A)$ and linearly independent. Hence $\{\vec{v}_1, \vec{v}_2\}$ is a basis.

Alternatively, we find a basis for $\ker(A)$ by using Gaussian elimination.

$$\begin{bmatrix} 3 & -1 & 1 & \vdots & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 & 1/3 & \vdots & 0 \end{bmatrix}.$$

We see that the second and third columns do not contain pivots, and so the asssociated unknown is free, so we get that the kernel is $\{x = s/3 - t/3, y = s, z = t\}$ or equivalently $\vec{x} = s\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$, where s and t are free. It follows that a basis is $\left\{\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

9. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{bmatrix}.$$

Let $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Find equations in b_1, b_2, b_3, b_4 so that the equation $A\vec{x} = \vec{b}$ can be solved. Find a basis of the image of A.

We use Gaussian elimination to find the equations in b_1, b_2, b_3, b_4 so that the equation $A\vec{x} = \vec{b}$ can be solved. Note that this is exactly the same as asking for the equations in b_1, b_2, b_3, b_4 so that $\vec{b}$ is in the image of A and exactly the same as asking for the equations in b_1, b_2, b_3, b_4 so that the linear system $A\vec{x} = \vec{b}$ is consistent.

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & b_1 \\ -1 & 2 & 0 & \vdots & b_2 \\ 1 & 1 & 3 & \vdots & b_3 \\ -2 & 1 & -3 & \vdots & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & \vdots & b_1 \\ 0 & 2 & 2 & \vdots & b_2 + b_1 \\ 0 & 1 & 1 & \vdots & b_3 - b_1 \\ 0 & 1 & 1 & \vdots & b_4 + 2b_1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & b_1 \\ 0 & 1 & 1 & \vdots & (b_2 + b_1)/2 \\ 0 & 1 & 1 & \vdots & b_3 - b_1 \\ 0 & 1 & 1 & \vdots & b_4 + 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & \vdots & b_1 \\ 0 & 1 & 1 & \vdots & (b_2 + b_1)/2 \\ 0 & 0 & 0 & \vdots & b_3 - b_1 - (b_2 + b_1)/2 \\ 0 & 0 & 0 & \vdots & b_4 + 2b_1 - (b_2 + b_1)/2 \end{bmatrix}$$

It follows that we need $b_3 - b_1 - (b_2 + b_1)/2 = 0$ and $b_4 + 2b_1 - (b_2 + b_1)/2 = 0$ for the system to be consistent. Simplifying the equations yield

$$\{2b_3 - 3b_1 - b_2 = 0, 2b_4 + 3b_1 - b_2 = 0\}.$$

The equations in the b_i define the image of A, so we can use this to construct a basis. Alternatively, we have computed rref(A). We see that the first 2 columns are pivot columns. This implies that the first two columns of A will form a basis for im(A), and so $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for im(A).

10. Let V, W be subspaces of $\mathbb{R}^n$. Assume that $V \subset W$ and that the dimension of V is equal to the dimension of W. Show $V = W$.

    We will prove this by contradiction. Suppose that $V \neq W$. Then there is a vector $\vec{v} \in W$ such that $\vec{v} \notin V$. Let S be a basis for V. Since $\vec{v} \notin V = \text{span}(S)$, the set $\tilde{S} = S \cup \{\vec{v}\}$ is linearly independent. This is impossible because

    $$\#\tilde{S} = \#S + 1 = \dim(V) + 1 = \dim(W) + 1.$$

    Therefore $V = W$.

11. Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}$. What are the possible values for the dimension of the kernel of T?

    The Rank-Nullity Theorem says that

    $$\dim(\text{im}(T)) + \dim(\ker(T)) = \dim(\mathbb{R}^5) = 5.$$

    Since the range is 1-dimensional, $\dim(\text{im}(T)) = 0$ or 1. The dimension of the domain is 5, so the Rank-Nullity Theorem implies that the dimension of the kernel is either 5 or 4 corresponding to these two cases. (Note: The dimension of the kernel is 5 only for the zero transformation.)