1. Write the system of equations as a matrix equation and find all solutions using Gauss elimination:

\[ \begin{align*}
\begin{array}{l}
x + 2y + 4z &= 0, \\
-x + 3y + z &= -5, \\
2x + y + 5z &= 3
\end{array}
\end{align*} \]

We see that this is a linear system with 3 equations in 3 unknowns. The matrix equation is

\[ A\vec{x} = \vec{b}, \]

where

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
-1 & 3 & 1 \\
2 & 1 & 5
\end{bmatrix}
\quad \text{and} \quad
\vec{b} = \begin{bmatrix}
0 \\
-5 \\
3
\end{bmatrix}.
\]

To solve this system, we form the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 4 : 0 \\
-1 & 3 & 1 : -5 \\
2 & 1 & 5 : 3
\end{bmatrix}
\]

and perform Gaussian elimination to get the coefficient matrix in reduced echelon form.

\[
\begin{align*}
&\begin{bmatrix}
1 & 2 & 4 : 0 \\
-1 & 3 & 1 : -5 \\
2 & 1 & 5 : 3
\end{bmatrix} \\
\rightarrow &\begin{bmatrix}
1 & 2 & 4 : 0 \\
0 & 5 & 5 : -5 \\
0 & -3 & -3 : 3
\end{bmatrix} \\
\rightarrow &\begin{bmatrix}
1 & 0 & 2 : 2 \\
0 & 1 & 1 : -1 \\
0 & 0 & 0 : 0
\end{bmatrix}
\end{align*}
\]

Since there is no pivot in the third column, the third unknown is free and the solutions are

\[ x = 2 - 2t, \quad y = -1 - t, \quad z = t \]

or equivalently \( \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \), where \( t \) is free.

2. What does it mean for a vector to be in the kernel of a matrix \( A \). Let \( A \) be the matrix

\[
\begin{bmatrix}
1 & 2 & 5 \\
-2 & 0 & -2 \\
3 & -1 & 1
\end{bmatrix}
\]

Is \( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \) an element of the kernel of \( A \)? Why?

A vector \( \vec{v} \) is in the kernel of \( A \) if \( A\vec{v} = \vec{0} \). To see that \( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \ker(A) \), we compute

\[
\begin{bmatrix}
1 & 2 & 5 \\
-2 & 0 & -2 \\
3 & -1 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 4 - 5 \\ -2 + 0 + 2 \\ 3 - 2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]
3. Define what it means for a set $s$ to be a basis of a subspace $V \subset \mathbb{R}^n$. Let
\[
A = \begin{bmatrix}
1 & 2 & 3 & -1 \\
-1 & 0 & 1 & -1 \\
-1 & 4 & 3 & -5
\end{bmatrix}.
\]
Give a set of vectors that span $\ker(A)$ and that are independent.

A set of vectors $s$ is a basis of $V$ if $V = \text{span}(s)$ and $s$ is linearly independent. To find a basis for $\ker(A)$, we use Gaussian elimination to compute the reduced echelon form of $A$.

\[
\begin{align*}
\begin{bmatrix}
1 & 2 & 3 & -1 \\
-1 & 0 & 1 & -1 \\
-1 & 4 & 3 & -5
\end{bmatrix} &\rightarrow \begin{bmatrix}
1 & 2 & 3 & -1 \\
0 & 2 & 4 & -2 \\
-1 & 4 & 3 & -5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 & -1 \\
0 & 6 & 6 & -6 \\
0 & 6 & 6 & -6
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & -1 \\
0 & 0 & -6 & 0
\end{bmatrix} &\rightarrow \begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\end{align*}
\]

We see that the 4th column has no pivot, and so $x_4$ is free. Then $\ker(A)$ is $x_1 = -t, x_2 = t, x_3 = 0, x_4 = t$ or equivalently $\vec{x} = t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Therefore a basis for $\ker(A)$ is
\[
\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

4. Let $A$ be a $n$ by $m$ matrix, so $A$ gives a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^m$. Assume that $A(\vec{x}_1) = A(\vec{x}_2)$. Show that $\vec{x}_1 - \vec{x}_2$ is in the kernel of $A$.

We compute
\[
A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{0}.
\]

5. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a vector of length 1. Let $A$ be a matrix whose effect on the plane is to reflect about the line through the origin and $\vec{u}$. Let $\vec{v} = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$. In terms of $\vec{u}$ and $\vec{v}$ what is $A\vec{u}$? what is $A\vec{v}$? Write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of $\vec{u}$ and $\vec{v}$. Use the answer to the previous question to compute $A\vec{e}_1$.

Notice that $\vec{v}$ is perpendicular to $\vec{u}$ (Check by computing dot product), and hence perpendicular to the line through the origin and $\vec{u}$. Since $A$ is reflection about this line, it follows that $A\vec{v} = -\vec{v}$. Since $\vec{u}$ is on the line, $A\vec{u} = \vec{u}$. We can use Gaussian
elimination on \[
\begin{bmatrix}
  u_1 & -u_1 & : & 1 \\
  u_2 & u_1 & : & 0
\end{bmatrix}
\] to express \( \vec{e}_1 \) as a linear combination of \( \vec{u} \) and \( \vec{v} \).

Alternatively, notice that \( u_1 \vec{u} - u_2 \vec{v} = \left( \frac{u_1^2 + u_2^2}{u_1u_2 - u_1u_2} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \), since \( \vec{u} \) has length 1.

Now compute
\[
A\vec{e}_1 = A(u_1 \vec{u} - u_2 \vec{v})
\]
\[
= u_1 \vec{u} + u_2 \vec{v}
\]
\[
= \left( \frac{u_1^2 - u_2^2}{2u_1u_2} \right).
\]

6. Solve the equation
\[
\begin{bmatrix}
  1 & 0 & -1 \\
  0 & 1 & 2 \\
  2 & 1 & -1
\end{bmatrix}
x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]
for \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) by find the inverse of the given matrix.

7. Compute the product \( AB \) of the two matrices \( A, B \) given below, if possible. If it is not possible say why it is not possible.

\[
A = \begin{bmatrix}
  1 & 2 \\
  -1 & 0 \\
  3 & -2
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
  -1 & 0 \\
  4 & 8
\end{bmatrix}
\]

The product matrix \( AB \) gives a function. What is the domain and what is the range of that function?

Since \( B \) gives a transformation \( \mathbb{R}^2 \to \mathbb{R}^2 \) and \( A \) gives a transformation \( \mathbb{R}^2 \to \mathbb{R}^3 \), only the composition \( AB \) makes sense and give a map from \( \mathbb{R}^2 \to \mathbb{R}^3 \). We compute the product
\[
AB = \begin{bmatrix}
  1 & 2 \\
  -1 & 0 \\
  3 & -2
\end{bmatrix} \begin{bmatrix}
  -1 & 0 \\
  4 & 8
\end{bmatrix} = \begin{bmatrix}
  7 & 16 \\
  1 & 0 \\
 -11 & -16
\end{bmatrix}
\]

8. Find a basis of the subspace of \( \mathbb{R}^3 \) defined by \( 3x - y + z = 0 \). What is the dimension of this subspace?

This subspace is the kernel of \( A = \begin{bmatrix} 3 & -1 & 1 \end{bmatrix} \). Since \( \text{rank}(A) = 1 \), the Rank-Nullity Theorem implies that the dimension of \( \ker(A) = 3 - 1 = 2 \). By inspection, \( \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \)
and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are in $\ker(A)$ and linearly independent. Hence $\{\vec{v}_1, \vec{v}_2\}$ is a basis.

Alternatively, we find a basis for $\ker(A)$ by using Gaussian elimination.

\[
\begin{bmatrix} 3 & -1 & 1 & : & 0 \\ -1 & 2 & 0 & : & 0 \\ 1 & 1 & 3 & : & 0 \\ -2 & 1 & -3 & : & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 0 & : & 0 \\ 1 & 1 & 3 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ -2 & 1 & -3 & : & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & : & b_1 \\ 0 & 2 & 2 & : & b_2 + b_1 \\ 0 & 1 & 1 & : & b_3 - b_1 \\ 0 & 1 & 1 & : & b_4 + 2b_1 \end{bmatrix}
\]

We see that the second and third columns do not contain pivots, and so the associated unknown is free, so we get that the kernel is $\{x = s/3 - t/3, y = s, z = t\}$ or equivalently $\vec{x} = s \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$, where $s$ and $t$ are free. It follows that a basis is $\left\{ \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

9. Consider the matrix

\[
A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{bmatrix}
\]

Let $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Find equations in $b_1, b_2, b_3, b_4$ so that the equation $A\vec{x} = \vec{b}$ can be solved. Find a basis of the image of $A$.

We use Gaussian elimination to find the equations in $b_1, b_2, b_3, b_4$ so that the equation $A\vec{x} = \vec{b}$ can be solved. Note that this is exactly the same as asking for the equations in $b_1, b_2, b_3, b_4$ so that $\vec{b}$ is in the image of $A$ and exactly the same as asking for the equations in $b_1, b_2, b_3, b_4$ so that the linear system $A\vec{x} = b$ is consistent.

\[
\begin{bmatrix} 1 & 0 & 2 & : & b_1 \\ -1 & 2 & 0 & : & b_2 \\ 1 & 1 & 3 & : & b_3 \\ -2 & 1 & -3 & : & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & : & b_1 \\ 0 & 2 & 2 & : & b_2 + b_1 \\ 0 & 1 & 1 & : & b_3 - b_1 \\ 0 & 1 & 1 & : & b_4 + 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & : & b_1 \\ 0 & 1 & 1 & : & (b_2 + b_1)/2 \\ 0 & 1 & 1 & : & b_3 - b_1 \\ 0 & 1 & 1 & : & b_4 + 2b_1 \end{bmatrix}
\]

It follows that we need $b_3 - b_1 - (b_2 + b_1)/2 = 0$ and $b_4 + 2b_1 - (b_2 + b_1)/2 = 0$ for the system to be consistent. Simplifying the equations yield $\{2b_3 - 3b_1 - b_2 = 0, 2b_4 + 3b_1 - b_2 = 0\}$. 


The equations in the \( b_i \) define the image of \( A \), so we can use this to construct a basis. Alternatively, we have computed \( \text{rref}(A) \). We see that the first 2 columns are pivot columns. This implies that the first two columns of \( A \) will form a basis for \( \text{im}(A) \), and so \[
\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\} \]
is a basis for \( \text{im}(A) \).

10. Let \( V, W \) be subspaces of \( \mathbb{R}^n \). Assume that \( V \subset W \) and that the dimension of \( V \) is equal to the dimension of \( W \). Show \( V = W \).

We will prove this by contradiction. Suppose that \( V \neq W \). Then there is a vector \( \vec{v} \in W \) such that \( \vec{v} \notin V \). Let \( S \) be a basis for \( V \). Since \( \vec{v} \notin V = \text{span}(S) \), the set \( \tilde{S} = S \cup \{ \vec{v} \} \) is linearly independent. This is impossible because\[
\#\tilde{S} = \#S + 1 = \dim(V) + 1 = \dim(W) + 1.
\]
Therefore \( V = W \).

11. Let \( T \) be a linear transformation from \( \mathbb{R}^5 \) to \( \mathbb{R} \). What are the possible values for the dimension of the kernel of \( T \)?

The Rank-Nullity Theorem says that\[
\dim(\text{im}(T)) + \dim(\text{ker}(T)) = \dim(\mathbb{R}^5) = 5.
\]
Since the range is 1-dimensional, \( \dim(\text{im}(T)) = 0 \) or 1. The dimension of the domain is 5, so the Rank-Nullity Theorem implies that the dimension of the kernel is either 5 or 4 corresponding to these two cases. (Note: The dimension of the kernel is 5 only for the zero transformation.)